Part I

Instructions.

- Check that this examination booklet has pages 1 through 6.
- This part of the examination consists of 10 multiple-choice questions. Each question is followed by four possible answers, at least one of which is correct. If more than one choice is correct, choose **only the best one**. Among the correct answers, the best answer is the one that implies (or includes) the other correct answer(s). **Indicate your chosen answer in the Answer Booklet by writing a, b, c or d against the relevant question number.**
- For each question, you will get 2 marks if you choose only the best answer. If you choose none of the answers, then you will get 0 for that question. **However, if you choose something other than the best answer or multiple answers, then you will get −2/3 mark for that question.**

You may begin now. Good luck!

QUESTION 1. Suppose a real valued function f is defined for all real numbers excepting 0, and satisfies the following condition: $f(xy) = f(x) + f(y)$ for all x, y in the domain. Consider the statements:

$$f(1) = f(-1) = 0 \tag{i}$$

$$f(x) = f(-x) \quad \text{for every } x \tag{ii}$$

(a) (i) is true and (ii) is false.

(b) (i) is false and (ii) is true.

(c) Both are true.

(d) Both are false.

QUESTION 2. Let f be a real-valued differentiable function defined for all $x \geq a$. Consider the function F defined by $F(x) = \int_a^x f(t)\,dt$. If f is increasing on any interval, then on that interval F is

(a) convex

(b) concave

(c) increasing

(d) decreasing

QUESTION 3. Consider the function $f : \Re \to \Re$ defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \in Q \\ 0, & \text{otherwise} \end{cases}$$

where Q is the set of rational numbers. Then, f is

(a) discontinuous at every $x \neq 0$.

(b) is continuous at all points in Q.

(c) continuous at multiple points.

(d) is discontinuous on a countable set of points.

The next four questions are based on the following data. *Consider a Society consisting of individuals. These individuals may belong to various sets called Clubs and Tribes. The collections of Clubs and Tribes satisfy the following rules:*

- *The entire Society is a Tribe.*
- *The empty subset of Society is also a Tribe.*
- *Given any collection of Tribes, the set of individuals who belong to each Tribe in that collection is also a Tribe.*
- *Given any two Tribes, the set of individuals who belong to at least one of these Tribes is also a Tribe.*
- *A set of individuals is called a Club if and only if the set of individuals not in it constitute a Tribe.*

QUESTION 4. The intersection of two Clubs is necessarily

(a) a Club

(b) a Tribe

(c) not a Club

(d) not a Tribe

QUESTION 5. The union of a collection of Clubs is necessarily

(a) not a Club

(b) not a Tribe

(c) a Club

(d) a Tribe

QUESTION 6. Which of the following statements is necessarily true?

(a) A set of individuals cannot be a Tribe and a Club.

(b) There are at least two sets of individuals that are both a Club and a Tribe.

(c) The union of a Club and a Tribe is a Tribe.

(d) The intersection of a Club and a Tribe is a Club.

QUESTION 7. Suppose we are given a Club and a Tribe. Then, the set of individuals who belong to the given Tribe but not to the given Club necessarily constitute

(a) a Club

(b) a Tribe

(c) neither a Club, nor a Tribe

(d) a Club and a Tribe

QUESTION 8. Two players, A and B, will play a best of seven table tennis match (i.e., the first to win 4 games will win the match, and the match will have at most 7 games). The two players are equally likely to win any of the games in the match. The probability that the match will end in 6 games is

(a) less than the probability that it will end in 7 games.

(b) equal to the probability that it will end in 7 games.

(c) greater than the probability that it will end in 7 games.

(d) None of (a), (b) or (c) is true.

The next two questions are based on the following information. *Suppose X and Y are two random variables. X can take values −1 and 1. Y can take integer values between 1 and 6. The following is the joint probability distribution of X and Y.*

	$Y = 1$	$Y = 2$	$Y = 3$	$Y = 4$	$Y = 5$	$Y = 6$
$X = -1$	0.1	a	0.3	0	0	0
$X = 1$	0	0	b	0.1	0.1	0.1

It is known that the expectations of the two random variables are $E(X) = -0.2$ and $E(Y) = 3.2$. Then

QUESTION 9. The value 'a' in the table equals

(a) 0

(b) 0.1

(c) 0.2

(d) 0.3

QUESTION 10. The value 'b' in the table equals

(a) 0

(b) 0.1

(c) 0.2

(d) 0.3

Part II

Instructions.

- Answer any four out of Questions 11, 12, 13, 14 and 15 in the Answer Booklet.
- Each question is worth 20 marks.

QUESTION 11. Let V be a vector space and let $\mathcal{L}(V, V)$ be the set of all linear transformations $T : V \to V$. Let $I \in \mathcal{L}(V, V)$ be the identity transformation. $P \in \mathcal{L}(V, V)$ is called a projector of V if (a) $V = \mathcal{R}(P) \oplus \mathcal{N}(P)$, and (b) $P(u + w) = u$ for all $u \in \mathcal{R}(P)$ and $w \in \mathcal{N}(P)$; where $\mathcal{R}(P)$ denotes the range space of P, $\mathcal{N}(P)$ denotes the null space of P and $\oplus$ denotes a direct sum. Prove the following statements.

(A) P is a projector of V if and only if it is idempotent, i.e., $P^2 = P$.

(B) If U is a vector space and $X : U \to V$ is a linear transformation with $\mathcal{R}(P) = \mathcal{R}(X)$, then P is a projector if and only if $PX = X$.

(C) P is a projector if and only if $I - P$ is a projector.

QUESTION 12. Let $r_1 < r_2$, $\tau \in [r_1, r_2]$, and for every $t \in [r_1, r_2]$, let $A(t)$ be an $n \times n$ real matrix and let $b(t) \in \Re^n$. Suppose, for every $\psi \in \Re^n$, there exists one and only one differentiable function $\phi : [r_1, r_2] \to \Re^n$ such that $D\phi(t) = A(t)\phi(t)$ for every $t \in [r_1, r_2]$ and $\phi(\tau) = \psi$.

(A) Show that the space of solutions of the ordinary differential equation $Dx(t) = A(t)x(t)$ is an n-dimensional vector space.

(B) A collection $\{\phi^1, \ldots, \phi^m\}$ of solutions of $Dx(t) = A(t)x(t)$ is linearly independent if and only if $\{\phi^1(t), \ldots, \phi^m(t)\} \subset \Re^n$ is linearly independent for every $t \in [r_1, r_2]$.

QUESTION 13. Consider a metric space (X, d) and a function $f : X \to \Re$. f is said to be lower semicontinuous at $x \in X$ if for every $\epsilon > 0$, there exists an open neighborhood $U(x) \in \mathcal{T}$ of x, such that $f(U(x)) \subset (f(x) - \epsilon, \infty)$. Prove the following statements.

(A) f is lower semicontinuous if and only if $f^{-1}((r, \infty))$ is open in (X, d) for every $r \in \Re$.

(B) If $\{f_i \mid i \in I\}$ is a family of lower semicontinuous functions $f_i : X \to \Re$, then the function $g : X \to \bar{\Re}$ defined by $g(x) = \sup\{f_i(x) \mid i \in I\}$ is lower semicontinuous.

(C) If $\{f_i \mid i \in I\}$ is a family of lower semicontinuous functions $f_i : X \to \Re$ and I is a finite set, then the function $g : X \to \bar{\Re}$ defined by $g(x) = \inf\{f_i(x) \mid i \in I\}$ is lower semicontinuous.

QUESTION 14. Let (X, d_X) and (Y, d_Y) be metric spaces and $F : X \to 2^Y$. Given $E \in 2^Y$, i.e., $E \subset Y$, define $F^+(E) = \{x \in X \mid F(x) \subset E\}$.

F is said to be upper hemicontinuous at $x \in X$ if for every open subset V of Y such that $F(x) \subset V$, there exists an open neighborhood U of x such that $U \subset F^+(V)$. F is said to be upper hemicontinuous on X if it is upper hemicontinuous at every $x \in X$.

(A) Show that F is upper hemicontinuous on X if and only if $F^+(E)$ is open in X for every E open in Y.

Consider metric spaces (X, d_X), (Y, d_Y) and (Z, d_Z), and mappings $F_1 : X \to 2^Y$ and $F_2 : Y \to 2^Z$. Define $F_2 \circ F_1 : X \to 2^Z$ by $F_2 \circ F_1(x) = \cup_{y \in F_1(x)} F_2(y)$ for $x \in X$.

(B) Show that if F_1 and F_2 are upper hemicontinuous, then $F_2 \circ F_1$ is upper hemicontinuous.

Consider a family of functions $\{F_i : X \to 2^Y \mid i \in I\}$. The union of these mappings is $\cup_{i \in I} F_i : X \to 2^Y$ defined by the formula $(\cup_{i \in I} F_i)(x) = \cup_{i \in I} F_i(x)$.

(C) Show that if F_i upper hemicontinuous for every $i \in I$ and I is finite, then $\cup_{i \in I} F_i$ is upper hemicontinuous.

QUESTION 15. (A) Let x be a Bernoulli random variable with p.d.f.

$$f(x) = \begin{cases} p^x(1-p)^{1-x}, & \text{if } x \in \{0, 1\} \\ 0, & \text{otherwise} \end{cases}$$

Let y be a Binomial random variable with p.d.f.

$$f(y) = \begin{cases} \binom{n}{y} p^y(1-p)^{1-y}, & \text{if } y \in \{0, 1, \ldots, n\} \\ 0, & \text{otherwise} \end{cases}$$

Derive the moment generating functions of x and y, and show that the latter is equal to the former raised to power n.

(B) Prove that linear functions of the form $\underset{\sim}{y} = \underset{\sim}{b} + B\underset{\sim}{x}$ are normal random vectors provided that $\underset{\sim}{x}$ is a normal random vector. Find $E(\underset{\sim}{y})$ and $V(\underset{\sim}{y})$. Prove that the normal random variables in $\underset{\sim}{y}$ are independent if and only if $V(\underset{\sim}{y})$ is a diagonal matrix.