

151

Delhi School of Economics
Department of Economics

Entrance Examination for M. A. Economics

Option B

June 24, 2006

Time. 3 hours

Maximum marks. 100

General instructions. Please read the following instructions carefully.

- Check that your examination has pages 1 to 6 and you have been given a blank Answer booklet for writing your answers. Do not start writing until instructed to do so by the invigilator.

- Fill in your Name and Roll Number on the small slip attached to the Answer booklet. Do not write this information anywhere else in this booklet.

- When you finish, hand in the Answer booklet to the invigilator.

- Do not disturb your neighbours at any time. Anyone engaging in illegal examination practices will be immediately evicted and that person's candidature will be cancelled.

152

SECTION I. Answer all questions (i.e. Questions 1, 2 and 3) in this section.

Question 1. Each of the ten parts of this question is followed by four possible answers, one of which is correct. Indicate the correct answer for each part in your answer booklet.

The following information applies to parts A, B and C of Question 1. Three players X , Y and Z take turns playing a game as follows. X and Y play in the first round. The winner plays Z in the second round, while the loser sits out. The winner of the second round plays the person who was sitting out. The game continues in this fashion, with the winner of the current round playing the next round with the person who sits out in the current round. The game ends when a player wins twice in succession; this player is declared the winner of the contest. For any of the rounds, assume that the two players playing the round each have a probability $1/2$ of winning the round, regardless of how the past rounds were won or lost.

A. The probability that X becomes the winner of the contest is

- (a) $5/14$
- (b) $1/2$
- (c) $3/7$
- (d) $7/16$

B. The probability that Z becomes the winner of the contest is

- (a) $1/7$
- (b) $1/5$
- (c) $1/8$
- (d) $2/7$

C. The probability that the game continues indefinitely, with no one winning twice in succession, is

- (a) $1/10^{23}$
- (b) 0
- (c) $1/2^{23}$
- (d) $1/2^{16}$

D. Amit has a box containing 6 red balls and 3 green balls. Amita has a box containing 4 red balls and 5 green balls. Amit randomly draws one ball from his box and puts it into Amita's box. Now Amita randomly draws one ball out of her box. What is the probability that the balls drawn by Amit and Amita were of different colours?

153

- (a) $1/3$
- (b) $2/15$
- (c) $4/15$
- (d) $7/15$

E. Two patients share a hospital room for two days. Suppose that, on any given day, a person independently picks up an airborne infection with probability $1/4$. An individual who is infected on the first day will certainly pass it to the other patient on the second day. Once contracted, the infection stays for at least two days. What is the probability that both patients have contracted the infection by the end of the second day?

- (a) $125/256$
- (b) $121/256$
- (c) $135/256$
- (d) $131/256$

F. A blood test detects a given disease with probability $8/10$ given that the tested person actually has the disease. With probability $2/10$, the test incorrectly shows the presence of the disease in a disease-free person. Suppose $1/10$ of the population has the disease. What is the probability that the person tested actually has the disease if the test indicates the presence of the disease?

- (a) 1
- (b) $9/13$
- (c) $4/13$
- (d) $7/13$

Answer G, H, I, and J using the following information. Consider a Society consists of individuals who may belong to various sets called Families and/or Gangs. The collections of Families and Gangs satisfy the following rules:

- The entire Society is a Family.
- The empty subset of Society is also a Family.
- Given a collection of Families, the set of individuals who belong to every Family in that collection is also a Family.
- Given any two Families, the set of individuals who belong to either of the two Families is also a Family.
- A set of individuals is called a Gang if and only if the set of individuals not in it constitute a Family.

154

G. The intersection of two Gangs is necessarily

- (a) a Family
- (b) a Gang
- (c) not a Family
- (d) not a Gang

H. The union of a collection of Gangs is necessarily

- (a) not a Family
- (b) not a Gang
- (c) a Family
- (d) a Gang

I. Which of the following statements is necessarily true?

- (a) A set of individuals cannot be a Gang and a Family.
- (b) There are at least two sets of individuals that are both a Family and a Gang.
- (c) The union of a Family and a Gang is a Gang.
- (d) The intersection of a Family and a Gang is a Family.

J. Suppose we are given a Family and a Gang. Then, the set of individuals who belong to the given Family but not to the given Gang necessarily constitute

- (a) a Family
- (b) a Gang
- (c) neither a Family, nor a Gang
- (d) a Family and a Gang

(20)

Question 2. Let x be a Bernoulli random variable with p.d.f.

$$f(x) = \begin{cases} p^x(1-p)^{1-x}, & \text{if } x \in \{0, 1\} \\ 0, & \text{otherwise} \end{cases}$$

Let y be a Binomial random variable with p.d.f.

$$f(y) = \begin{cases} \binom{n}{y} p^y (1-p)^{1-y}, & \text{if } y \in \{0, 1, \dots, n\} \\ 0, & \text{otherwise} \end{cases}$$

Derive the moment generating functions of x and y , and show that the latter is equal to the former raised to power n .

(10)

155

Question 3. Prove that linear functions of the form $\underline{y} = \underline{b} + B\underline{x}$ are normal random vectors provided that \underline{x} is a normal random vector. Find $E(\underline{y})$ and $V(\underline{y})$. Prove that the normal random variables in \underline{y} are independent if and only if $V(\underline{y})$ is a diagonal matrix.

(10)

SECTION II. Answer any three out of Questions 4, 5, 6 and 7.

Question 4. Consider a metric space (X, d) and a function $f : X \rightarrow \mathbb{R}$. f is said to be lower semicontinuous at $x \in X$ if for every $\epsilon > 0$, there exists an open neighborhood $U(x) \in \mathcal{T}$ of x , such that $f(U(x)) \subset (f(x) - \epsilon, \infty)$. Prove the following statements.

(A) f is lower semicontinuous if and only if $f^{-1}((r, \infty))$ is open in (X, d) for every $r \in \mathbb{R}$.

(B) If $\{f_i \mid i \in I\}$ is a family of lower semicontinuous functions $f_i : X \rightarrow \mathbb{R}$, then the function $g : X \rightarrow \mathbb{R}$ defined by $g(x) = \sup\{f_i(x) \mid i \in I\}$ is lower semicontinuous.

(C) If $\{f_i \mid i \in I\}$ is a family of lower semicontinuous functions $f_i : X \rightarrow \mathbb{R}$ and I is a finite set, then the function $g : X \rightarrow \mathbb{R}$ defined by $g(x) = \inf\{f_i(x) \mid i \in I\}$ is lower semicontinuous.

(20)

Question 5. Let (X, d_X) and (Y, d_Y) be metric spaces and $F : X \rightarrow 2^Y$. Given $E \in 2^Y$, i.e., $E \subset Y$, define $F^+(E) = \{x \in X \mid F(x) \subset E\}$.

F is said to be upper hemicontinuous at $x \in X$ if for every open subset V of Y such that $F(x) \subset V$, there exists an open neighborhood U of x such that $U \subset F^+(V)$. F is said to be upper hemicontinuous on X if it is upper hemicontinuous at every $x \in X$.

(A) Show that F is upper hemicontinuous on X if and only if $F^+(E)$ is open in X for every E open in Y .

Consider metric spaces (X, d_X) , (Y, d_Y) and (Z, d_Z) , and mappings $F_1 : X \rightarrow 2^Y$ and $F_2 : Y \rightarrow 2^Z$. Define $F_2 \circ F_1 : X \rightarrow 2^Z$ by $F_2 \circ F_1(x) = \bigcup_{y \in F_1(x)} F_2(y)$ for $x \in X$.

(B) Show that if F_1 and F_2 are upper hemicontinuous, then $F_2 \circ F_1$ is upper hemicontinuous.

Consider a family of functions $\{F_i : X \rightarrow 2^Y \mid i \in I\}$. The union of these mappings is $\bigcup_{i \in I} F_i : X \rightarrow 2^Y$ defined by the formula $(\bigcup_{i \in I} F_i)(x) = \bigcup_{i \in I} F_i(x)$.

(C) Show that if F_i upper hemicontinuous for every $i \in I$ and I is finite, then $\bigcup_{i \in I} F_i$ is upper hemicontinuous.

(20)

Question 6. Consider vector spaces V and W . Let $\mathcal{L}(V, W)$ denote the space of linear transformations from V to W . Let $A \in \mathcal{L}(V, W)$.

$A^L \in \mathcal{L}(W, V)$ is said to be a left inverse of A if $A^L A = I \in \mathcal{L}(V, V)$. $A^R \in \mathcal{L}(W, V)$ is said to be a right inverse of A if $AA^R = I \in \mathcal{L}(W, W)$. $A^- \in \mathcal{L}(W, V)$ is said to be a generalized inverse of A if $AA^-A = A$. $\rho(A)$, $\nu(A)$ and $\mathcal{R}(A)$ denote the rank, nullity and range space, respectively, of a linear transformation A . Prove the following facts.

$$(A) \rho(A^-) \geq \rho(A).$$

$$(B) \rho(AA^-) = \rho(A^-A) = \rho(A).$$

$$(C) \nu(AA^-) = \nu(A^-A) = \nu(A).$$

$$(D) AA^- \text{ projects } W \text{ on } \mathcal{R}(A) \text{ and } A^-A \text{ projects } V \text{ on } \mathcal{R}(A^-A).$$

Let U be a vector space and $B \in \mathcal{L}(W, U)$. Prove the following facts.

$$(E) \text{ If } \rho(BA) = \rho(B), \text{ then } A(BA)^- = B^-.$$

$$(F) \text{ If } \rho(BA) = \rho(A), \text{ then } (BA)^-B = A^-.$$

$$(G) \text{ If } \rho(BA) = \rho(A), \text{ then } A(BA)^-B \text{ projects } W \text{ on } \mathcal{R}(A).$$

(20)

Question 7. Consider the Euclidean metric space $(\mathbb{R}^n, \|\cdot\|)$. Let $X \subset \mathbb{R}^n$ and $f : X \rightarrow \mathbb{R}$. X is said to be a convex set if for every $x, y \in X$ and $t \in (0, 1)$, we have $tx + (1-t)y \in X$. f is said to be a convex function at $x \in X$ if for every $y \in X$ and $t \in (0, 1)$, $tx + (1-t)y \in X$ implies $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$. f is said to be a convex function if it is a convex function at every $x \in X$. Prove the following statements.

(A) If X is a convex set, then $f : X \rightarrow \mathbb{R}$ is a convex function if and only if $\{(x, y) \in X \times \mathbb{R} \mid f(x) \leq y\}$ is a convex set.

(B) If X is open in \mathbb{R}^n and f is convex and differentiable at $x \in X$, then

$$f(y) - f(x) \geq Df(x) \cdot (y - x)$$

for every $y \in X$, where $Df(x)$ denotes the derivative of f at x .

(C) If X is open in \mathbb{R}^n and f is convex and twice differentiable at $x \in X$, then $D^2f(x)$ is positive semidefinite.

(20)