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Delhi School of Economics
Department of Economics

Entrance Examination for M. A. Economics
Option B
June 25, 2005

Time. 3 hours **Maximum marks.** 100

Instructions. Please read the following instructions carefully.

- Check that your examination has pages 1 to 6 and you have been given a blank Answer booklet for writing your answers. Do **not** start writing until instructed to do so by the invigilator.
- Fill in your Name and Roll Number on the small slip attached to the Answer booklet. Do **not** write this information anywhere else on this booklet.
- When you finish, hand in the Answer booklet to the invigilator.
- Do not disturb your neighbours at any time. **Anyone engaging in illegal examination practices will be immediately evicted and that person's candidature will be cancelled.**

Do not write below this line.

This space is for official use only.

Marks tally

Question	Marks
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Fictitious Roll Number

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Instruction I. *Question 1 is compulsory.*

Question 1. Indicate the correct choices for A to J.

Answer A, B, C and D using the following information. Consider a Society consists of individuals. These individuals may belong to various sets called Clubs and/or Tribes. The collections of Clubs and Tribes satisfy the following rules:

- The entire Society is a Club.
- The empty subset of Society is also a Club.
- Given a collection of Clubs, the set of individuals who belong to at least one of these Clubs is also a Club.
- Given any two Clubs, the set of individuals who belong to both Clubs is also a Club.
- A set of individuals is called a Tribe if and only if the set of individuals not in it constitute a Club.

A. The union of two Tribes is necessarily

(a) a Club

(b) a Tribe

(c) not a Club

(d) not a Tribe

B. The intersection of a collection of Tribes is necessarily

(a) not a Club

(b) not a Tribe

(c) a Club

(d) a Tribe

C. Which of the following statements is necessarily true?

(a) A set of individuals cannot be a Tribe and a Club.

(b) There are at least two sets of individuals that are both a Club and a Tribe.

(c) The union of a collection of Tribes is a Tribe.

(d) The intersection of a collection of Clubs is a Club.

D. Suppose we are given a Club and a Tribe. Then, the set of individuals who belong to the given Club but not to the given Tribe necessarily constitute

(a) a Club

(b) a Tribe

(c) neither a Club, nor a Tribe

(d) a Club and a Tribe

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E. A number of mathematicians in the middle of the 20th century contributed to a series of books published in the name of a fictitious mathematician called Bourbaki. Suppose a sociological critic of science asserts "There exists a book by Bourbaki such that every chapter in that book contains a theorem whose validity depends on the reader's gender." If this assertion is false, which of the following assertions must be true?

(a) Every book by Bourbaki contains a chapter such that the validity of some theorem in that chapter is independent of the reader's gender.

(b) Every chapter in every book by Bourbaki contains a theorem whose validity is independent of the reader's gender.

(c) There exists a book by Bourbaki such that every chapter in it contains a theorem whose validity is independent of the reader's gender.

(d) Every book by Bourbaki contains a chapter such that the validity of all the theorems in it is independent of the reader's gender.

F. Suppose X and Y are independent random variables with standard Normal distributions. The probability of $X < -1$ is some $p \in (0, 1)$. What is the probability of the event: $X^2 > 1$ and $Y^3 < -1$?

(a) $3p$
(b) p^2
(c) $2p^2$
(d) $3p^2$

G. There are three identical boxes, each with two drawers. Box A contains a gold coin in each drawer. Box B contains a silver coin in each drawer. Box C contains a gold coin in one drawer and a silver coin in another drawer. A box is chosen, a drawer opened and a gold coin is found. What is the probability that the chosen box is A?

(a) 2/3
(b) 1/3
(c) 1/2
(d) 3/4

H. A random variable has outcomes Success and Failure with probabilities 3/4 and 1/4 respectively. A gambler observes the sequence of outcomes of this variable and receives a prize of 2^n if n is the first time that Success occurs. What is the expected value of the gambler's prize?

(a) 1
(b) 2
(c) 3

$\frac{3}{4}\times 2 + \frac{1}{4}\times\frac{3}{4}\times 2^2 + \frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}\times 2^3 + \frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}\times 2^4 + \frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}\times 2^5$

$= \frac{3}{4}\left(2 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right) = \frac{3}{4}(2) + \frac{3}{4}\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{3}{2} + \frac{3}{4}\times 2 = 3$

(d) 4

Answer I and J using the following information. Suppose n voters use the following procedure to find their leader. Each voter is given a coin that has probability 1/2 of falling Heads and probability 1/2 of falling Tails. Each voter tosses his/her coin. A person is chosen leader if his/her toss outcome is different from that of the other $n-1$ persons' tosses. This procedure is iterated until a leader is determined.

I. The probability of finding a leader in the k-th iteration of the procedure is

(a) $2^{1-n}(1-2^{1-n})^{k-1}$

(b) $n2^{1-n}(1-n2^{1-n})^{k-1}$

(c) $n2^{-n}(1-n2^{-n})^{k-1}$

(d) $2^{-n}(1-2^{-n})^{k-1}$

J. If $n=3$, then the probability of finding a leader in up to two iterations is

(a) 7/16

(b) 15/16

(c) 39/64

(d) 15/64

Instruction II. *Answer any two out of Questions 2, 3 and 4.*

Question 2. Let $\{F_n \mid n \in \mathcal{N}\}$ be a family of nonempty closed subsets of the Euclidean space $\Re^p$ with F_1 bounded and $F_{n+1} \subset F_n$ for every $n \in \mathcal{N}$. Show that $\cap_{n\in\mathcal{N}} F_n \neq \emptyset$.

Question 3. Consider a metric space (X,d).

(A) Show that if x is a convergent sequence in (X,d), then it is a Cauchy sequence in (X,d).

Suppose (X,d) is complete. Let $f : X \to X$ be a contraction, i.e., there exists $\beta \in (0,1)$ such that, for all $x,y \in X$, we have $d(f(x),f(y)) \leq \beta d(x,y)$. Let $x_0 \in X$. Define the sequence x inductively by the formula $x_n = f(x_{n-1})$ for $n \in \mathcal{N}$.

(B) Show that x is a Cauchy sequence, and therefore convergent.

(C) Show that a limit point of x, say α, is a fixed point of f, i.e., $\alpha = f(\alpha)$.

(D) Show that f has a unique fixed point.

Question 4. A set $X \subset \Re^n$ is called a cone in $\Re^n$ if $x \in X$ implies $tx \in X$ for every $t \in \Re_+$. Given a cone X in $\Re^n$, a function $f : X \to \Re$ is said to be positive homogeneous of degree $k \in \Re$ if $f(tx) = t^k f(x)$ for every $x \in X$ and every $t \in \Re_{++}$.

(A) Let X be the interior of a cone in $\Re^n$ and let $f : X \to \Re$ be differentiable on X. Show that f is positive homogeneous of degree k if and only if $kf(x) = Df(x).x$ for every $x \in X$.

(Notation: $Df(x)$ is the derivative of f at x and $Df(x).x$ is the (unitary) inner product of $Df(x)$ and x.)

(B) Let X be the interior of a cone in $\Re^n$ and let $f : X \to \Re$ be differentiable on X. Show that, if f is positive homogeneous of degree k, then the partial derivative $D_i f$ is homogeneous of degree $k-1$ for $i = 1, \dots, n$.

Instruction III. *Answer either Question 5 or 6.*

Question 5. Suppose $\mathcal{U}$ is a vector space with U as the set of vectors and $\mathcal{V}$ is a vector space with V as the set of vectors. Let $\mathcal{L}(\mathcal{U}, \mathcal{V})$ be the space of linear transformations from U to V. Let $I \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ denote the identity transformation on V. Given $P \in \mathcal{L}(\mathcal{V}, \mathcal{V})$, let $\mathcal{R}_P$ be the range space of P and let $\mathcal{N}_P$ be the null space of P.

$P \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ is called a projector of $\mathcal{V}$ if

(a) $\mathcal{V} = \mathcal{R}_P \oplus \mathcal{N}_P$, and

(b) for every $u \in \mathcal{R}_P$ and $w \in \mathcal{N}_P$, we have $P(u + w) = u$.

Prove the following propositions for $P \in \mathcal{L}(\mathcal{V}, \mathcal{V})$.

(A) P is a projector of $\mathcal{V}$ if and only if it is idempotent, i.e., $P^2 = P$.

(B) If $\mathcal{U}$ is a vector space and $X \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ with $\mathcal{R}_P = \mathcal{R}_X$, then P is a projector if and only if $PX = X$.

(C) P is a projector if and only if $I - P$ is a projector.

Question 6. Suppose $\mathcal{V}$ is a vector space with V as the set of vectors and $\langle .,. \rangle : V \times V \to \Re$ is an inner product on $\mathcal{V}$. Let $\mathcal{W}$ be a subspace of $\mathcal{V}$ with the set of vectors W. Let $\mathcal{O}(\mathcal{W}) = \cap_{x \in W} \{y \in V \mid \langle x, y \rangle = 0\}$.

(A) Show that $\mathcal{O}(\mathcal{W})$ is a subspace of $\mathcal{V}$.

$\{c_1, \dots, c_r\} \subset V$ is called an orthonormal set if for all $i, j \in \{1, \dots, r\}$,

$$\langle c_i, c_j \rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

A basis $\{c_1, \dots, c_r\}$ for $\mathcal{V}$ is called an orthonormal basis if $\{c_1, \dots, c_r\}$ is an orthonormal set.

(B) Show that, if $\{c_1, \dots, c_r\}$ is an orthonormal basis for $\mathcal{V}$, then $y = \sum_{i=1}^{r} \langle y, c_i \rangle c_i$ for every $y \in V$.

Fact: If $\mathcal{V}$ is finite dimensional, then every orthonormal subset of V can be extended to an orthonormal basis for $\mathcal{V}$.

(C) Use the Fact to show that, if $\mathcal{V}$ is finite dimensional and $\mathcal{W}$ is a subspace of $\mathcal{V}$, then $\mathcal{W} \oplus \mathcal{O}(\mathcal{W}) = \mathcal{V}$.

Instruction IV. *Answer either Question 7 or 8.*

Question 7. (A) A number X and a sequence of numbers $\{Y_n \mid n \in \mathcal{N}\}$ are drawn from the uniform distribution on $[0, 1]$. Let $N = \inf\{n \in \mathcal{N} \mid Y_n > X\}$. The player conducting these draws receives the prize Rs. $(N - 1)$. Calculate the expected value of this prize.

(B) Suppose X and Y are independent real-valued random variables whose distributions have densities f and g respectively. Let $Z = X + Y$. Derive the density of the distribution of Z if

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$g(y) = \begin{cases} 1, & \text{if } y \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

Question 8. (A) In an opinion poll it is assumed that an unknown proportion p of the people are in favour of a proposed new law and a proportion $1-p$ are against it. A sample of n people is taken to obtain their opinion. The proportion $\bar{p}$ in favour in the sample is taken as an estimate of p. Using the Central Limit Theorem, determine how large a sample will ensure that the estimate will, with probability 0.95, be correct to within 0.01.

(B) Suppose X and Y are independent real-valued random variables whose distributions have densities f and g respectively. Let $Z = X + Y$. Derive the density of the distribution of Z if

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$g(y) = \begin{cases} \mu e^{-\mu y}, & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases}$$