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Delhi School of Economics
Department of Economics

Entrance Examination for M. A. Economics
Option B
June 26, 2004

Time. 3 hours **Maximum marks.** 100

Instructions. Please read the following instructions carefully.

- Check that your examination has pages 1 to 5 and you have been given a blank Answer booklet for writing your answers. Do **not** start writing until instructed to do so by the invigilator.
- Fill in your Name and Roll Number on the small slip attached to the Answer booklet. Do **not** write this information anywhere else on this booklet.
- The examination has 10 questions, each worth 20 marks. Answer any 5 of these questions.
- When you finish, hand in the Answer booklet to the invigilator.
- Do not disturb your neighbours at any time. **Anyone engaging in illegal examination practices will be immediately evicted and that person's candidature will be cancelled.**

Do not write below this line.

This space is for official use only.

Marks tally

Question	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

Question 1. Given vectors $x, y \in \Re^n$, let $\langle x, y \rangle$ denote an inner product of x and y.

(A) Prove that $\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$.

(B) Use (A) to show that $\langle x + y, x + y \rangle^{1/2} \leq \langle x, x \rangle^{1/2} + \langle y, y \rangle^{1/2}$.

Question 2. Suppose $\Re$ is endowed with the Euclidean metric.

(A) Prove that a function $f : \Re \to \Re$ is continuous at $x \in \Re$ if and only if the sequence $(f(x_n))$ converges to $f(x)$ for every sequence (x_n) converging to x.

(B) Suppose $f : \Re \to \Re$ is defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ irrational} \\ 1 - x, & \text{if } x \text{ rational} \end{cases}$$

Show that f is continuous at $x = 1/2$ and discontinuous elsewhere.

Question 3. (A) Consider the function $f : \Re^2 \to \Re$ defined by

$$f(x, y) = \begin{cases} 0, & \text{if } (x, y) = (0, 0) \\ xy^2/(x^2 + y^2), & \text{if } (x, y) \neq (0, 0) \end{cases}$$

Show that the partial derivative of f at $(0, 0)$ with respect to any direction vector $u = (a, b)$; denoted by $D_u f(0, 0)$, exists and that

$$D_u f(0, 0) = ab^2/(a^2 + b^2)$$

if $(a, b) \neq (0, 0)$. Also show that f is continuous but not differentiable at $(0, 0)$.

(B) Consider the function $f : \Re^5 \to \Re^2$ defined by

$$f(u, v, w, x, y) = (uy + vx + w + x^2, uvw + x + y + 1)$$

and note that $f(2, 1, 0, -1, 0) = (0, 0)$.

(*i*) Show that we can solve $f(u, v, w, x, y) = (0, 0)$ for (x, y) in terms of (u, v, w) in a neighborhood of $(2, 1, 0)$.

(*ii*) If $(x, y) = \phi(u, v, w)$ is the solution for (*i*), then show that the derivative of ϕ at $(2, 1, 0)$ is

$$D\phi(2, 1, 0) = \frac{1}{3} \begin{bmatrix} 0 & -1 & -3 \\ 0 & 1 & -3 \end{bmatrix}$$

Question 4. A set $X \subset \Re^n$ is said to be convex if $tx + (1 - t)y \in X$ for all $x, y \in X$ and $t \in (0, 1)$. Given a convex set $X \subset \Re^n$, a function $f : X \to \Re$ is said to be concave if $f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y)$ for all $x, y \in X$ and $t \in (0, 1)$.

(A) Show that $f : \Re^n \to \Re$ is concave if and only if

$$\sum_{i=1}^{r} t_i f(x_i) \leq f\left(\sum_{i=1}^{r} t_i x_i\right)$$

for every positive integer r, for all $x_1, \ldots, x_r \in \Re^n$, and all $t_1, \ldots, t_r \in (0,1)$ such that $\sum_{i=1}^{r} t_i = 1$.

(B) Use (A) to show that

$$\prod_{i=1}^{r} x_i^{t_i} = x_1^{t_1} x_2^{t_2} \ldots x_r^{t_r} \leq \sum_{i=1}^{r} t_i x_i$$

for all non-negative $x_1, \ldots, x_r \in \Re$ and $t_1, \ldots, t_r \in (0,1)$ with $\sum_{i=1}^{r} t_i = 1$.

(C) Show that a function $f : \Re^n \to \Re$ is concave if and only if the set

$$\{(x, r) \in \Re^n \times \Re \mid f(x) \leq r\}$$

is convex.

Question 5. Let U and V be vector subspaces of $\Re^n$. Define

$$U + V = \{u + v \mid u \in U \ \&\ v \in V\}$$

The following is a fact: $\dim(U+V) + \dim(U \cap V) = \dim(U) + \dim(V)$, where $\dim(.)$ refers to the dimension of the relevant vector space.

Let S and T be linear transformations from $\Re^n$ to $\Re^n$. Use the above-mentioned fact to prove

(A) $\rho(T + S) \leq \rho(T) + \rho(S)$, and

(B) $\nu(T + S) \geq \nu(T) + \nu(S) - n$,

where $\rho(.)$ refers to the rank of the relevant transformation and $\nu(.)$ refers to the nullity of the relevant transformation.

Question 6. (A) Suppose $\{\alpha^1, \alpha^2, \ldots, \alpha^n\}$ is a basis for $\Re^n$ and $c_i \in \Re$ is non-zero for $i = 1, \ldots, n$. Prove that $\{c_1\alpha^1, c_2\alpha^2, \ldots, c_n\alpha^n\}$ is a basis for $\Re^n$.

(B) Consider an $n \times n$ matrix A. Prove that if A is either symmetric or skew-symmetric, then A^2 is symmetric.

(C) An $n \times n$ matrix A is called idempotent if $A^2 = A$. Prove that if $A \neq I$ and A is idempotent, then A is singular.

(D) Consider the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & -1 & 1 \end{bmatrix}$$

For what values of k are all the eigenvalues of this matrix real? For what values are they all real and negative?

Question 7. (A) Consider an examinee answering a multiple-choice examination. For a particular question with 4 choices, the probability that the examinee knows the answer is 1/3, the probability that she makes the correct choice given that she knows the answer is 1, and the probability that she makes the correct choice given that she does not know the answer is 1/4. What is the probability that she knew the answer given that she has made the correct choice?

(B) Each of four economists is asked to prescribe one out of four economic policies. Each economist is equally likely to prescribe any of the four different policies. What is the probability that each of the economists prescribes a different policy?

(C) There are three identical boxes, each with two drawers. Box A contains a gold coin in each drawer. Box B contains a silver coin in each drawer. Box C contains a gold coin in one drawer and a silver coin in another drawer. A box is chosen, a drawer opened and a gold coin is found. What is the probability that the chosen box is A?

(D) 100 outcomes of a random variable were recorded. The sample mean is 40 and sample standard deviation is 5.1. It is discovered that one observation was wrongly recorded as 50 instead of 40. What are the correct mean and standard deviation?

(E) Suppose X and Y are independent random variables with standard Normal distributions. The probability of $X < -1$ is some $p \in (0, 1)$. What is the probability of the event: $X^2 > 1$ and $Y^3 > 1$?

Question 8. (A) The random variables $X_1, \ldots, X_n$ are independent draws from a continuous uniform distribution with support $[0, \theta]$. Derive a Method of Moments and Maximum Likelihood estimators of θ. Your research assistant (RA) has constructed new random variables Y_i such that

$$Y_i = \begin{cases} 0, & \text{if } X_i \le k \\ 1, & \text{if } X_i > k \end{cases}$$

where k is a constant chosen by the RA and known to you. What are the Maximum Likelihood and Method of Moments estimators in this case? (Assume $k < \theta$.)

(B) Suppose you are interested in estimating the fish in a pond. You release 100 tagged fish into the pond and you keep catching and releasing fish until you get one that

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is tagged. Say, this happens at N. You estimate the number of fish as $P = 100 \times N$. Is this estimator unbiased? What is its variance?

Question 9. Consider a Bernoulli random variable:

$$X_i = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p \end{cases}$$

You observe the outcome of two Bernoulli trials and want to test $H_0: \ p = 0$ against $H_A: \ p = 0.5$. Use the Neyman-Pearson lemma to determine the most powerful test of H_0 versus H_A. What are the type 1 and type 2 errors for your test?

Question 10. (A) Derive the Laplace transform (i.e., moment generating function) of a Gaussian (i.e., Normal) random variable with mean μ and variance σ^2.

(B) Suppose X is a random variable such that $\ln X$ is Normal with mean μ and variance σ^2. Find the mean and variance of X.

(C) Suppose X_1 and X_2 are independent Normal random variables. Suppose X_i has mean μ_i and variance σ_i^2. Show that $X_1 + X_2$ is Normal with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.