Test Codes: QEA and QEB (Both are of 'Descriptive' type)
(Junior Research Fellowship in Quantitative Economics)

The candidates for Junior Research Fellowship in Quantitative Economics are required to take two descriptive-type tests - QEA in the forenoon session and QEB in the afternoon session. While the questions in QEA are on Mathematics, those in QEB are on Economics.

Syllabus for QEA

1. Permutations and combinations.
2. Theory of quadratic and cubic equations; Elementary set theory.
3. Matrix algebra, rank and inverse of matrices, linear equations, determinants, and eigenvalues.
4. Functions of one and two variables: Limits, continuity, differentiation, applications, integration of elementary functions, and definite integrals.
5. Constrained and unconstrained optimization, convexity of sets ,and concavity and convexity of functions.
6. Elements of probability theory, discrete and continuous random variables, expectation and variance, joint conditional and marginal distributions, and distribution of function of a random variable.

Syllabus for QEB

1. Theory of consumer behaviour; Theory of production; Market structure; General equilibrium and welfare economics; International trade and finance; Public economics.
2. Macroeconomic theories of income determination, rational expectations, Phillips Curve, neo-classical growth model, and inequality.
3. Game Theory: Normal and extensive forms, and Nash and sub-game perfect equilibrium.
4. Multiple Regression Model: Least squares and maximum likelihood methods of estimation, heteroscedasticity, autocorrelation , multicollinearity, specification bias, exogeneity, and instrumental variables.
5. Time Series Analysis: Stationarity, trend, seasonality, unit root tests, and ARMA model.

Sample Questions for QEA

1. The derivative of x^2 with respect to *x* is 2*x*. However, let us write x^2 as the sum of *x x*'s, and then take the derivative. That is, let $f(x)=x+x+...+x$ (x times), then

$$f'(x)=\frac{d}{dx}[x]+\frac{d}{dx}[x]+...+\frac{d}{dx}[x] \ (x \text{ times})=1+1+...+1 \ (x \text{ times})=x.$$

This argument appears to show that the derivative of x^2 with respect to *x* is actually *x*. Where is the fallacy? Explain clearly.

2. Find the number of nonnegative integer solutions of the equation $x_1+x_2+x_3=10$.

3. Suppose *A* is an $n \times n$ symmetric *positive definite* matrix and *B* is an $n \times m$ matrix with *rank* (*B*) = *m*. Prove that $B^T AB$ is also a symmetric *positive definite* matrix. (B^T denotes the transpose of matrix *B*.)

4. Let $f:[1,4] \to \Re$ be a continuous function and $f(1)=f(4)$. Prove that there exists at least one point $\xi \in [1,4]$ such that $f(\xi)=f(\xi+1.5)$.

5. Suppose f_n, $n=1,2,3,...$, is a sequence of real valued functions defined on a set $S \subseteq \Re$. We say that the sequence f_n converges pointwise to a function *f* defined on *S* if $\lim_{n \to \infty} f_n(x)=f(x), \ \forall x \in S$.

   Let $f_n(x)=\left(1-|x|\right)^n, \ \forall x \in (-1,1)$. Find the function *f* to which this sequence converges pointwise.

6. Consider the following definitions.

   Convex Combination: A vector $y \in \Re^n$ is said to be a convex combination of the vectors $x^1, x^2, ..., x^m \in \Re^n$ if there exist *m* non-negative real numbers $\theta_1, \theta_2, ..., \theta_m$ such that (i) $\sum_{i=1}^{m} \theta_i = 1$, and (ii) $y=\sum_{i=1}^{m} \theta_i x^i$.

   Convex Set: A set $S \subset \Re^n$ is a *convex set* if for every two vectors in *S*, all convex combinations of these two vectors are also in *S*.

Use the method of induction to prove that if a set $S \subset \Re^n$ is convex, then for any integer $m > 1$ and for any *m* vectors in *S*, every convex combination of these *m* vectors is in *S*.

7. Let *X*, *Y* be random variables such that $(X,Y) \in \{(x,y) | 0 \le x \le 3,\ 0 \le y \le 4\}$ always. The joint cumulative density function (c.d.f) of *X* and *Y* in this rectangle is

$$F(x,y) = \frac{xy(x^2 + y)}{156}.$$

Find

(a) $P(1 \le X \le 2 \text{ and } 1 \le Y \le 2)$,

(b) the cumulative density function of *Y*,

(c) the joint probability density function of *X* and *Y*,

(d) $P(Y \le X)$.

8. (a) Suppose a surveillance system has a 99% chance of correctly identifying a terrorist and a 99.9% chance of correctly identifying someone who is not a terrorist. Suppose there are 1000 terrorists in an adult male population of 300 million, and one of these 300 million men is randomly selected, scrutinised by the system, and identified as a terrorist. Then the probability that he is actually a terrorist is closest to

(i) 0.99, (ii) 2/3, (iii) 1/3, (iv) 1/300.

Justify your answer.

(b) A class contains 10 boys and 15 girls. 8 students are to be selected at random from the class without replacement. Let *X* be the number of boys selected and *Y* the number of girls selected. Find $E(X - Y)$. Be sure to give the reason for each step in your answer.

9. Let $A \subset \Re^2$ be open and $f : A \to \Re$ be twice continuously differentiable. Consider the problem of maximizing $f(x,y;a)$ with respect to $(x,y) \in \Re^2$, where $a \in \Re$ is a *parameter* for the maximization problem.

Given any $a \in \Re$, suppose $(x^*(a), y^*(a))$ is a solution to the maximization problem. Provide (with a clear explanation) a *sufficient* condition for $x^*(a)$ and $y^*(a)$ to be *continuously differentiable* functions of *a*.

10. Recall the following version of Lagrange Theorem for optimization. *Let* $A \subset \Re^2$ *be open, and* $f : A \to \Re$ *and* $g : A \to \Re$ *be continuously differentiable functions on A. Suppose* (x^*, y^*) *is a point of local maximum or minimum of* $f(x,y)$ *subject to the constraint* $g(x,y) = 0$. *Suppose further that* $\left(\frac{\partial g}{\partial x}(x^*, y^*), \frac{\partial g}{\partial y}(x^*, y^*)\right) \neq (0,0)$. *Then there exists* $\lambda^* \in \Re$ *such that* $\frac{\partial f}{\partial x}(x^*, y^*) = \lambda^* \frac{\partial g}{\partial x}(x^*, y^*)$, *and* $\frac{\partial f}{\partial y}(x^*, y^*) = \lambda^* \frac{\partial g}{\partial y}(x^*, y^*)$.

Let *f* and *g* be functions on $\Re^2$ defined respectively by

$$f(x,y) = \tfrac{1}{3}x^3 - \tfrac{3}{2}y^2 + 2x,$$

and

$$g(x,y) = x - y.$$

Consider the problems of *maximizing* and *minimizing* *f* on the set $C = \{(x,y) \mid g(x,y) = 0\}$.

Examine whether you can apply the Lagrange Theorem to solve these *maximization* and *minimization* problems. If yes, find the solutions. If not, provide clear explanations.

Sample Questions for QEB

1. Let $\Re^3_{++} = \{x = (x_1, x_2, x_3) \in \Re^3_+ : x_1 > 0,\ x_2 > 0,\ x_3 > 0\}$. Suppose that the consumption set is $\Re^3_+$ and the initial endowment is w = ($\bar{x} = 2;\ \bar{y} = 5;\ \bar{z} = 7$). Find, with a clear explanation, the demand set $D(p)$ for any price $p \in \Re^3_{++}$ if the preference relation on $\Re^3_+$ is represented by a utility function

   a. $U(x, y, z) = \min\{x, y, z\}$, and

   b. $U(x, y, z) = \min\{x, y\}$.

2. Consider a monopolist facing two kinds of consumers, each with utility function $\theta_i V(q) - T$ if they consume q units, and pay T, and zero otherwise. Let $V(q) = \frac{1-(1-q)^2}{2}$. Let $\theta_2 > \theta_1$, with there being λ consumers of type 1 and $1-\lambda$ consumers of type 2. The monopolist's production cost is cq.

   a. What is the demand function of a consumer facing a price p? What is the consumers' surplus?

   b. Suppose the monopolist can use two-part tariffs consisting of a fixed fee F and a marginal fee of p on every unit consumed. What is the optimal two-part tariff under perfect discrimination?

   c. With a single price, and no fixed fee, what is the monopolist's optimal price assuming both goods are sold, and he does not know which consumer is of which type?

   d. What is the optimal two-part tariff when the monopolist cannot discriminate between consumers?

   e. Show that the price under the optimal two-part tariff is more efficient compared to that under one-part tariff.

3. Consider a market for used cars where each buyer wants to purchase exactly one used car if at all. In case he does not buy a car, his utility is zero. There are two kinds of used cars, good quality, called peaches, and bad quality, called lemons. Let there be 100 lemons with valuation 0 for both buyers and sellers, whereas there are 100

peaches with valuations of 100 for the buyers, and 60 for the sellers. There are 500 buyers, who do not know which car is of which type.

a. Explain what type of trade should take place in the efficient outcome.

b. Compute the market equilibrium when the buyers know which car is which type. Is the outcome efficient? Explain clearly.

c. Now consider the case with asymmetric information: the sellers know the type of their cars but the buyers do not. Explain the market outcome under asymmetric information. Explain clearly whether the market outcome will be efficient.

4. (a) Let $G = \langle N, S_1, \ldots, S_n, \pi_1, \ldots, \pi_n \rangle$ be a game in normal form where $N = \{1, 2, \ldots, n\}$ is the set of players, S_i and $\pi_i : S_1 \times S_2 \times \ldots \times S_n \rightarrow \Re$ are the strategy sets and payoff functions for player i, $i = 1, 2, \ldots, n$.

(i) When is a strategy $s_i \in S_i$ a strictly dominated strategy for player i?

(ii) Define a Nash equilibrium (in pure strategies).

(b) Consider the two-person game where $S_1 = S_2 = [0,1]$,

$\pi_1(s_1, s_2) = 3s_1s_2 - 2s_1 - 2s_2 + 2$, $\pi_2(s_1, s_2) = -4s_1s_2 + 2s_1 + s_2$.

Compute the best-response correspondences of the two-players and Nash equilibria (pure strategies) of the game.

5. There are two cities S and T. A total of 4000 cars need to go from S to T. There are two paths from S to T. Path 1 goes via city A – we will denote this as $S \rightarrow A \rightarrow T$. Path 2 goes via city B – we will denote this as $S \rightarrow B \rightarrow T$. Note that roads go in one direction.

On Path 1, the road from S to A takes $\frac{T}{100}$ units of time, where T is the number of cars travelling on this road. On the other hand, the road from A to T takes 45 units of time, irrespective of the number of cars travelling on this road.

On Path 2, the road from S to B takes 45 units of time, irrespective of the number of cars travelling on this road. On the other hand, the road from B to T takes $\frac{T}{100}$ units of time, where T is the number of cars travelling on this road.

Suppose the utility of every car is the negative of the amount of time spent in going from S to T.

a. What is a Pareto optimal way for the 4000 cars to travel from S to T?

b. Suppose each car (driver) is strategic in choosing his path and all cars choose their paths simultaneously. Describe a pure strategy Nash equilibrium of this game.

The planner decided to build another road to travel from A to B (you cannot travel from B to A on this road). The travel time from A to B is zero (irrespective of the number of cars traveling on this road). So, now there are three paths from S to T: Path 1 is $S \to A \to T$, Path 2 is $S \to B \to T$, and Path 3 is $S \to A \to B \to T$.

c. Suppose each car (driver) is strategic in choosing his path and all cars choose their paths simultaneously. Does the Nash equilibrium that you found when the road from A to B did not exist still a Nash equilibrium now? Describe a pure strategy Nash equilibrium of this game.

d. What conclusions can be drawn from this Nash equilibrium about whether the planner should build the road from A to B?

6. Consider an economy existing only for two periods, 1 and 2. This economy produces a single good, with the representative individual having a utility function given by

$$U = \ln(C_1) + \ln(l_1) + \delta[\ln(C_2) + \ln(l_2)]$$

where C_1 and C_2 are consumption of the good in periods 1 and 2, respectively, l_1 and l_2 are leisure time enjoyed by the agent in periods 1 and 2, and $0 < \delta < 1$ is the discount factor. This economy is a small open one facing a constant rate of interest r in the world capital market.

Each agent is endowed with one unit of time in each period which he can spend working or enjoying as leisure. The agent can produce an output y_t of the good in period t using the production function $y_t = 1 - l_t;\ t = 1, 2$. The representative agent's flow budget constraint in each period is given by

$$b_1 = (1 - l_1) - C_1(1 + \tau_1),$$

$$b_2 = (1 - l_2) + (1 + r)b_1 - C_2(1 + \tau_2),$$

where τ_1 and τ_2 are taxes on consumption imposed by the government and b_1 and b_2 are the net foreign assets at the end of periods 1 and 2, respectively.

Assume that $\delta(1 + r) = 1$.

(a) Set up the agent's utility maximization problem.

(b) Show that $l_1 = l_2$.

(c) Show that consumption in period 2 is higher relative to period 1 if $\tau_1 > \tau_2$.

7. Suppose a household lives for two periods, 1 and 2, and has no initial wealth. Assume for simplicity that this household has only one member. The household incomes in the two periods are given by $y_1 = w_1(1-l_1)$ and $y_2 = w_2(1-l_2)$, where w_1 and w_2 are real wages in periods 1 and 2, and l_1 and l_2 are leisure time enjoyed in periods 1 and 2. The household can borrow and save at the constant interest rate *r*. The household's utility function given by

$$U = \ln(C_1) + \frac{\beta(l_1)^{1-\gamma}}{1-\gamma} + \delta\left[\ln(C_2) + \frac{\beta(l_2)^{1-\gamma}}{1-\gamma}\right]$$

where $\beta > 0$, $\gamma > 0$, C_1 and C_2 are consumption of the good in periods 1 and 2, and $0 < \delta < 1$ is the discount factor.

(a) Write down the household's budget constraint.

(b) Set up the household's utility maximization problem.

(c) Derive how the relative demand for leisure, $\frac{l_1}{l_2}$, responds to changes in the relative wage, $\frac{w_2}{w_1}$. Calculating the elasticity of the change in relative demand for leisure with respect to the change in relative wage, explain how this responsiveness depends on γ.

(d) Derive how the relative demand for leisure, $\frac{l_1}{l_2}$, responds to changes in the gross interest rate, (1+*r*). Calculating the elasticity of the change in relative demand for leisure with respect to the change in gross interest rate, explain how this responsiveness depends on γ.

8. (a) Consider the standard Solow model with a given savings ratio (*s*), a given rate of growth of population (*n*), no depreciation, and a constant returns to scale production technology: $Y(t) = F(K(t), L(t))$.

(i) Derive the fundamental differential equation of the Solow model:

$$\dot{k}(t) = s \cdot f(k(t)) - nk(t),$$

where $k(t)=\dfrac{K(t)}{L(t)}$ and $f(k(t))$ is the production function in intensive form.

(ii) Explain the concepts of *golden rule of capital accumulation* and *dynamic efficiency* in the context of the above Solow model.

(b) Consider the above Solow model with the following modification: the wage-earners consume their entire income and save nothing, while the interest-earners save their entire income and consume nothing. The aggregate savings so generated is automatically invested in physical capital formation, which augments the stock of physical capital over time: $\dot{K}(t)=S(t)$.

Derive the differential equation for $k(t)$ for this modified model. Explain whether the steady-state level of capital in this model achieves the golden rule of capital accumulation.

9. Consider a Solow-Swan growth model augmented with human capital in which the constant returns to scale production function is given by

$$Y(t)=\left[K(t)\right]^{\alpha}\left[H(t)\right]^{1-\alpha},\ 0<\alpha<1,$$

and where K and H evolve according to

$$\dot{K}(t)=s_K Y(t),\ 0<s_K<1,$$

$$\dot{H}(t)=s_H Y(t),\ 0<s_H<1.$$

Here $K(t)$ is the aggregate capital stock and $H(t)$ is the stock of human capital.

(a) Show that regardless of the initial levels of K and H, the ratio $\dfrac{K}{H}$ converges to the same balanced growth path level, $\dfrac{K^*}{H^*}$.

(b) Once $\dfrac{K}{H}$ has converged to $\dfrac{K^*}{H^*}$, what are the growth rates of K, H, and Y?

(c) How does the growth rate of Y on the balanced growth path depend on s_K and s_H? Compare your answer to the effect of savings rate in the standard Solow growth model.

(d) Suppose $\frac{K}{H}$ starts off at a level that is smaller than $\frac{K^*}{H^*}$. Is the initial growth rate of *Y* greater than, less than, or equal to its growth rate on the balanced growth path? Explain clearly.

10. $X_1, \ldots\ldots X_n$ are independent random variables distributed identically as Uniform (*a*, *b*) where *a* and *b* are unknown parameters and *a* < *b*.

(a) Find the method of moments estimators for *a* and *b*.

(b) Find the maximum likelihood estimators for *a* and *b*.

11. A cross sectional survey of 260 households was carried out to learn about the determinants of out of pocket medical expenditures for households. Apart from information on expenditures on medical expenditures, the survey also collected information about household income, demographic information (average age of household members, proportion of females), access to nearest heath facilities. Summary of the data for medical expenditures yielded the following information:

Mean = 20,000, Median = 1000, Minimum Value = 100, Maximum Value = 100000.

Suppose you want to evaluate the determinants of medical expenditures using multivariate regressions.

(a) Argue why taking a logarithmic transformation of your dependent variable may be advisable while running an OLS (ordinary least squares) regression.

(b) If you run an OLS regression with the given set of variables, will your coefficients be consistent? Argue why or why not.

(c) Irrespective of your answer in part (b), suppose you run the regression

$$\ln(Y) = \delta x + \varepsilon,$$

where □ is a vector of covariates, $E(\varepsilon) = 0$, $E(x'\varepsilon) = 0$, and $\varepsilon \sim N(0, \sigma_\varepsilon^2)$.

(i) What is the impact of a change in a unit value of x_i on $E(Y|x)$?

(ii) Suppose you are told that ε is heteroscedastic, i.e., the variance is $\sigma_\varepsilon^2(x_i)$. How does it affect what can be learnt from the sign of the estimated δ?

12. Consider estimating the effect of personal computer ownership (PC) on performance in the first year of college (colmarks).

(a) Suppose you also have data on performance in the college entrance exam (ent_exam) and high school score (high_school). All of this information is obtained from a random survey of second year college students. Suppose you postulate the following model for each student *i* in your dataset:

$$colmarks_i = \beta_0 + \beta_1 \, high\_school_i + \beta_2 \, ent\_exam_i + \beta_3 \, PC_i + u_i,$$

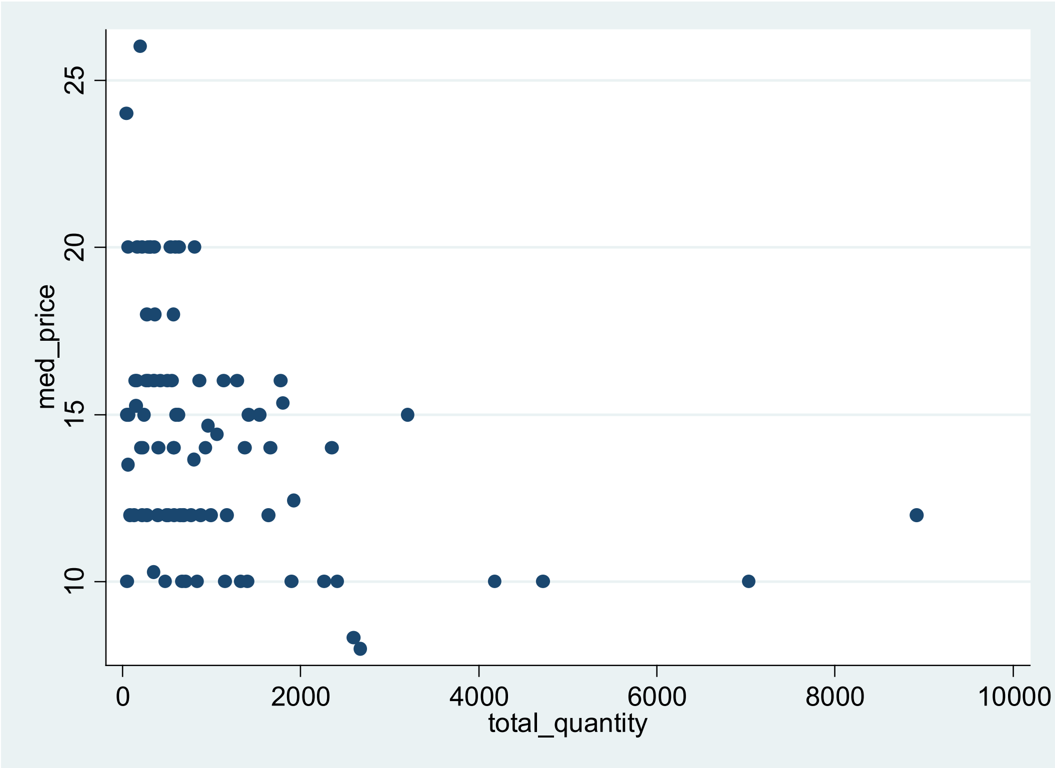
and estimate the model by OLS. Compute the probability limit of the coefficient on PC.

(b) In another study of the same question, the researchers designed the survey in the following manner. They first selected a random sample of students without personal computers. A random subset of these students was provided a PC in the first year while the others received it in the second year. Once again the researchers estimate by OLS the model:

$$colmarks_i = \beta_0 + \beta_1 \, high\_school_i + \beta_2 \, ent\_exam_i + \beta_3 \, PC_i + u_i.$$

Compute the probability limit of the coefficient on PC.

(c) The figure below provides the scatter plot of prices and quantities of potatoes using the National Sample Survey data on Consumption Expenditure Survey.



Each data point plotted represents a market m, that is, each point refers to the total quantity purchased/sold in the market (total_quantity: q_m) and median price (med_price: p_m) for a unit of quantity in the market (say per kg.).

Suppose you want to estimate the demand curve for potatoes and run an OLS bivariate regression:

$$q_m = a + bp_m + \varepsilon_m, \; m = 1, 2, \ldots, M.$$

(i) Is the OLS estimator of b consistent? Explain why or why not.

(ii) Suppose the input costs for producing potatoes varies across markets. Can data on input costs be useful to you for demand estimation?

13. (a) Check if the following time series $\{x_t\}$ is (weakly) stationary:

$$x_t - 1.6\, x_{t-1} + 0.6\, x_{t-2} = a_t,$$

where $\{a_t\}$ is white noise with zero mean and variance σ_a^2 .

Would the conclusion remain the same if the coefficient of x_{t-2} is replaced by 0.89? Give explanations for your answer.

(b) Obtain the autocorrelation function (ACF), in terms of the parameters involved, of the following special ARMA(1,2) process given by

$$x_t - \phi\, x_{t-1} = a_t - \theta a_{t-2},$$

where $\{a_t\}$ is white noise with zero mean and variance σ_a^2 , $|\phi| < 1$ and $|\theta| < 1$. State the similarities, if any, between the ACF of this process and that of an AR (1) process.