Test Code : RE I / RE II (Both Descriptive types)
(Junior Research Fellowship in Quantitative Economics)

The candidates for Junior Research Fellowship in Quantitative Economics are required to take two descriptive type tests - RE I (Mathematics) in the forenoon session and RE II (Economics) in the afternoon session.

Syllabus for RE I

1. Permutations and combinations. 2. Theory of quadratic equations. 3. Elementary set theory; functions and relations; matrices and determinants. 4. Convergence of sequences and series. 5. Functions of one and several variables: limits, continuity, differentiation, applications, integration of elementary functions, definite integrals. 6. Constrained and unconstrained optimization, convexity of sets and concavity and convexity of functions. 7. Elements of probability theory, discrete and continuous random variables, expectation and variance, joint conditional and marginal distributions, distributions of functions of random variables.

Syllabus for RE II

1. Theory of consumer behaviour; theory of production; market structure; general equilibrium and welfare economics; international trade and finance; public economics; information economics. 2. Macroeconomic theories of income determination, rational expectations, Phillips curve; Neoclassical growth model; inequality. 3. Game Theory: normal and extensive forms, Nash and sub-game perfect equilibrium. 4. Basic statistical inference; regression analysis (including heteroscedasticity, autocorrelation and multicollinearity), least squares and maximum likelihood estimation, specification bias, endogeneity and exogeneity, instrumental variables. 5. Modern time-series analysis.

Sample Questions for RE I

1. (a) In a pack of cards there are 52 cards: cards of 13 denominations belonging to each of 4 suits (clubs, diamonds, hearts and spades). 5 cards are drawn from a pack of 52 cards. What is the probability of getting 3 cards of one denomination and 2 cards of another denomination?

   (b) Suppose we need to infer the occurrence of a rare disease in a population of N people. If the blood of every person is tested, obviously N tests will have to be done. An alternate strategy is to pool all the blood samples into groups of k (Mix the blood samples of k people. Assume N and k are such that $\frac{N}{k}$ is an integer). Each pooled sample is then tested. If it is negative, all k people in the group are concluded to be free from the disease. If it is positive, each person in the group must be retested, ultimately requiring $k + 1$ tests being done on that particular group. Suppose the probability of any one person having a positive test is p. What will be the expected number of tests under the pooling plan?

2. (a) Suppose $S = \{1, 2, \ldots, n\}$, where $n \geq 2$. Suppose numbers a and b are chosen with replacement from S with equal probability. If $a = b$ then an amount x is given as reward; otherwise an amount y is penalized. What is the expected profit/loss from this game when n is finite? What is the expected profit/loss as $n \to \infty$?

   (b) Suppose n real numbers are drawn at random from the uniform distribution over the interval $[0, 1]$. For $x \in [0, 1]$, what is the probability that the second highest number drawn is $\leq x$?

3. A set of vectors x^1, x^2, ..., x^m (here $x^i = (x_1^i, x_2^i, ..., x_n^i) \in \mathbb{R}^n$ is a vector) is *linearly dependent* if there exist numbers $\lambda_1, \lambda_2, ..., \lambda_m$, *not all zero*, such that

$$\lambda_1 x^1 + \lambda_2 x^2 + ... + \lambda_m x^m = 0 \text{ (the zero vector in } \mathbb{R}^n).$$

A set of vectors is called *linearly independent* if the vectors are *not* linearly dependent.

Let x^1, x^2, ..., x^n be a set of linearly independent vectors in $\mathbb{R}^n$. Let y^1, y^2, ..., y^n be a set of vectors defined by

$$y^1 = x^1 + x^2, y^2 = x^2 + x^3, ..., y^{n-1} = x^{n-1} + x^n, y^n = x^n + x^1.$$

Is the set of vectors $\{y^1, y^2, ..., y^n\}$ linearly dependent? Explain clearly.

4. (a) Let A be an $m \times n$ matrix. Define

$$\text{cone } (A) = \left\{ b \in \mathbb{R}^m : Ax = b \text{ for some } x \in \mathbb{R}_+^n \right\}.$$

[Note that $x \in \mathbb{R}_+^n$ implies $x_1 \geq 0$, $x_2 \geq 0$, ..., $x_n \geq 0$.]

(i) Draw cone (A) when $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$.

(ii) Show that for any $m \times n$ matrix A, cone (A) is a convex set.

(b) Give an example of a set $S \subseteq \mathbb{R}^n$ which satisfies both the properties below:

(i) For every $x, y \in S$ we have $\frac{x+y}{2} \in S$;

(ii) The set S is not convex.

5. (a) Prove that there is a *unique* function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying both the properties below:

   (i) $f(x, y)$ is non-decreasing in x and non-decreasing in y;

   (ii) $f(x, x) = c$ for all $x \in \mathbb{R}$, where c is a constant.

   How will your answer change if "$f(x, y)$ is non-*decreasing* in x and non-*decreasing* in y" is replaced by "$f(x, y)$ is non-*increasing* in x and non-*increasing* in y".

   (b) Find the function $f : \mathbb{R}^2 \to \mathbb{R}$ which satisfies both the properties below:

   (i) $f(x + h, y) = f(x, y) + h(1 + y)$ for all $x, y, h \in \mathbb{R}$;

   (ii) $f(0, y) = 0$ for all $y \in \mathbb{R}$.

6. (a) One solution to the system of equations

$$\begin{aligned} x + 2y + z &= 5, \\ 3x^2yz &= 12, \end{aligned}$$

   is ($x = 2,\ y = 1,\ z = 1$). Classify the three variables into *exogenous* ones and *endogenous* ones in a neighbourhood of ($x = 2,\ y = 1,\ z = 1$) so that if one varies the exogenous variables near their original values and plug these new values into the given system of equations then one can find unique values of the endogenous variables that still satisfy the system.

   (b) Find the solution(s) to the following problem.

$$\begin{aligned} &Minimize_{\{x,y\}}\ px + qy, \\ &\text{subject to} \\ &\qquad ax^\alpha + by^\beta = M, \\ &\qquad x \geq 0,\ y \geq 0, \end{aligned}$$

   where $a > 0$, $b > 0$, $p > 0$, $q > 0$, $M > 0$ and $\alpha > 1$ and $\beta > 1$.

7. (a) Let

$$f(x) = e^x,\ g(x) = \begin{cases} x^2, & x < \dfrac{1}{2} \\ \dfrac{1}{4}, & x \geq \dfrac{1}{2} \end{cases} \quad \text{and } h(x) = g(f(x)).$$

Check if $h'(x)$ at $x = \dfrac{1}{2}$ exists.

(b) Let $\{a_n\}$ be the sequence defined by $a_n = \dfrac{9^n}{n!}$.

Find the limit of this sequence as $n \to \infty$.

Find also the maximum value of this sequence.

8. Let A be a convex subset of $\mathbb{R}^2$, $f : A \to \mathbb{R}$ be a *concave* function and $g : A \to \mathbb{R}$ be a *convex* function. Consider the maximization problem:

$$\left.\begin{array}{ll} \textit{Maximize} & f(x,y) \\ \text{subject to} & (x,y) \in C_b \equiv \{(x,y) \in A : g(x,y) \leq b\}. \end{array}\right\} \text{(P)}$$

For any fixed $b \in \mathbb{R}$, let $Z(b)$ denote the set of $(x,y) \in C_b$ that are global maximizers of f on C_b. For any $b \in \mathbb{R}$, let $V(b)$ denote the maximal value of the objective function f in problem (P).

(a) Prove that $Z(b)$ is a convex set.

(b) Let $b_3 = \lambda b_1 + (1-\lambda) b_2$, where $0 \leq \lambda \leq 1$, and let $(x_i, y_i) \in Z(b_i)$, $i = 1, 2, 3$.
Prove that $(\lambda \cdot (x_1, y_1) + (1-\lambda) \cdot (x_2, y_2)) \in C_{b_3}$.

(c) Use part (b) to prove that $V(b)$ is a *concave* function of b.

9. Let f be a twice continuously differentiable function on an open interval (a,b) in $\mathbb{R}$. Suppose that $f(c) = \dfrac{f(a)+f(b)}{2}$, where $c = \dfrac{a+b}{2}$. Using

the Mean Value Theorem (or otherwise) prove that there exists a real number $\xi \in (a, b)$ such that $f''(\xi) = 0$.

10. A consumer's utility function for sugar (S) and bread (B) is given by

$$u = (x_S)^{\alpha} (x_B)^{1-\alpha}, \ 0 < \alpha < 1.$$

A rationing scheme is in place so that the consumer needs *both* money and ration coupons to purchase bread and sugar. The consumer has Rs. 1,000 and the per unit prices of sugar (S) and bread (B) are $P_S =$ Rs. 10 and $P_B =$ Rs. 5, respectively. He also has 1000 ration coupons and must give 5 per unit of sugar and 10 per unit of bread.

   (a) Set up the consumer's utility maximization problem and draw the constraint set clearly.

   (b) Find out, with a clear explanation, the solutions to the utility maximization problem for appropriate ranges of the parameter α, and draw appropriate diagrams to illustrate your solutions. [Check in particular that the solution depends on the following 3 ranges of the parameter α: (i) $0 < \alpha \leq \frac{1}{3}$, (ii) $\frac{1}{3} < \alpha < \frac{2}{3}$, and (iii) $\frac{2}{3} \leq \alpha < 1$. Your diagram should distinguish these 3 parameter ranges clearly.]

Sample Questions for RE II

1. Consider the normal form game played by two players as shown in Table 1. In each cell of Table 1, the first number denotes the payoff of Player 1 and the second number denotes the payoff of Player 2.

   This game is played in periods $t = 1, 2$. Both players observe the actions of each other at the end of period 1. Payoff of every player at the end of period 2 is the sum of the payoff in periods 1 and 2. For example, if player 1 chooses U and D in periods 1 and 2 respectively and player 2 chooses L and R in periods 1 and 2 respectively, then the payoff of player 1 is $1 + 2 = 3$ and that of player 2 is $1 + 2 = 3$.

(a) Represent the game in extensive form.

(b) How many strategies does each player have in this extensive form game?

(c) What are the subgame perfect equilibria of this game?

	L	R
U	$1,1$	$0,3$
D	$3,0$	$2,2$

Table 1: A Normal Form Game

2. Consider an exchange economy with two agents and two commodities. Assume that agent i's utility function is given by

$$u_i(x_i, y_i) = \theta_i x_i^{\frac{1}{2}} + y_i$$

where x_i and y_i are the amounts of the two goods consumed by agent i and $\theta_i > 0$. Suppose agent 1 has one unit of good x while agent 2 has one unit of good 2.

(a) What is the set of Pareto-optimal allocations in this economy? Draw a diagram illustrating your answer.

(b) Compute the Walrasian equilibrium in the case where $\theta_1 = 2$ and $\theta_2 = 3$.

3. An economy has n consumers. Each consumer belongs to one of the two possible types, type 1 and type 2. Consumers have preferences over a private good x, and a non-excludable public good G. The utility of a representative agent of type i is

$$u_i = \ln x + \beta_i \ln G,$$

with $\beta_1 = 1$, $\beta_2 \in [0, 1)$.

Type 1 and 2 consumers respectively have an endowment of 1 unit and 0.5 units of the private good. Each consumer j contributes an amount g_j from his endowment for the production of the public good. One unit of the private good can be costlessly transformed into one unit of the public good, and vice versa. Hence the amount of public good produced is given by $G = \sum_{j=1}^{n} g_j$.

In a symmetric Nash equilibrium in which all consumers of the same type contribute the same amount for the public good, what will be the total production of the public good?

4. (a) Consider a duopoly with quantity competition and assume that quantities are strategic substitutes, that is, the marginal profitability of a firm is decreasing in the output of the other firm. Show that industry output under Stackelberg equilibrium is larger than that under Cournot equilibrium.

(b) Suppose that there are $n \geq 2$ firms out of which k firms ($0 \leq k \leq n$) join a cartel. The cartel acts as Stackelberg leader and the $n - k$ non-members play as independent followers. The inverse market demand function is $P(X) = 1 - X$, where $X = \sum_{i=1}^{n} x_i$, and the unit cost of production is zero for each firm. Derive the expressions for payoffs for each of the cartel member and the non-member firms in equilibrium.

5. "Anticipated macro policy changes have no impact on the time path of output." Defend or refute. [You must give formal reasoning along with intuitions.]

6. Consider an overlapping generations model of Tibet where the only difference with respect to the standard overlapping generations model is that an agent may choose to join a monastery in their old age. Let $I_t \in \{0, 1\}$ denote an indicator function, where $I_t = 1$ indicates a decision by agent t to join a monastery, and $I_t = 0$ indicates the decision not to join. A monk is not allowed to carry any of his income or savings from youth to the monastery; instead he enjoys a utility $x \in \mathbb{R}$ from leading a spiritual life (x is exogenous). The utility of agents born in period $t = 0, 1, 2...$ is as follows:

$$\begin{aligned} U_t &= \log(c_{1t}) + \beta \log(c_{2t+1}), \text{ if } I_t = 0, \\ U_t &= \log(c_{1t}) + x, \text{ if } I_t = 1. \end{aligned}$$

While young, agents face the period constraint, $c_{1t} + s_t = w_t$,where c_{1t} denotes consumption in youth, s_t denotes savings, and w_t denotes the wage rate. When old, agents just consume their savings from youth adjusted for the gross return on capital, R_{t+1}. That is, when old, the constraint is given by $c_{2t+1} = R_{t+1}s_t$.

(a) Consider the case $I_t = 0$. Derive the optimal savings (s_t), optimal consumption in youth (c_{1t}), and optimal second period consumption (c_{2t+1}).

(b) Substitute these decision rules back into the utility function and show that the indirect utility from $I_t = 0$ is given by

$$V_{0t} = \log(\frac{w_t}{1+\beta}) + \beta \log(\frac{w_t \beta R_{t+1}}{1+\beta}).$$

(c) Now consider the case, $I_t = 1$. What is the optimal savings and optimal level of first period consumption, c_{1t}? Show – in this case – that the indirect utility is given by

$$V_{1t} = \log w_t + \beta x.$$

(d) Derive a condition on x that ensures that $V_{0t} \geq V_{1t}$.

(e) What do individuals need to not join the monastery after the first period in terms the above condition? Answer this question with reference to w_t and R_{t+1} and explain this in one sentence intuitively.

(f) Suppose that the production function of the Tibetan economy is given by $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$.

Derive 1) the wage rate in terms of $k_t = \frac{K_t}{L_t}$, and 2) assuming $\delta = 1$, the expression for R_{t+1} (note that $R_{t+1} = r_{t+1} + 1 - \delta$).

Suppose only $p < 1$ fraction of workers (where p is exogenous) choose not to join the monastery after the first period, so that the capital market clearing condition is given by $k_{t+1} = ps_t$.

Derive the law of motion of capital. Is the steady state unique? What happens when p increases? Illustrate your answer graphically.

7. Consider a version of the Solow growth model with the standard assumptions: constant population growth rate, $\dfrac{\dot{L}(t)}{L(t)} = n > 0$; constant savings rate, $s(t) = sY(t)$, $s \in (0,1)$, and the constant rate of depreciation of capital, $\theta \in [0,1]$. The only difference with the standard Solow model is that the production function, $Y(t) = F(K(t), L(t))$, satisfies the property of *diminishing returns to scale.*

(a) Write down the time path of labour, $L(t)$.

(b) Derive the intensive form production function, $y(t) = f(k(t))$, where $y(t) = \dfrac{Y(t)}{L(t)}$, and $k(t) = \dfrac{K(t)}{L(t)}$.

(c) Using the time path of labour, $L(t)$, and the intensive form production function, $y(t) = f(k(t))$, derive the expression for the capital accumulation equation, $\dfrac{\dot{k}(t)}{k(t)}$.

(d) Explain the possibility of the existence of a steady-state equilibrium for this formulation of the Solow model.

8. (a) Consider an autoregressive process of order 2

$$x_t = \Phi_1 x_{t-1} + \Phi_2 x_{t-2} + a_t$$

where $\{a_t\}$ is a white noise process with mean zero and variance σ^2. Suppose that the following sample moments of $\{x_t\}$, based on 100 observations, were obtained for this process:

$$\upsilon_0 = 6.06,\ r_1 = 0.68,\ \text{and } r_2 = 0.61$$

where υ_0, r_1 and r_2 stand for the sample variance and sample autocorrelations of lag 1 and 2, respectively.

Find the estimates of the parameters Φ_1, Φ_2 and σ^2.

(b) The following estimated equation was obtained for testing stationarity of the monthly time series $\{x_t\}$ of fuel consumption in the state of Michigan, USA, by the augmented Dickey-Fuller (ADF) test.

$$\begin{array}{ccccccccccc} x_t & = & 0.827 & + & 0.025t & - & 0.028x_{t-1} & + & \sum_{j=1}^{4} \hat{\delta}_j \Delta x_{t-j} & + & \hat{a}_t \\ & & (0.231) & & (0.011) & & (0.012) & & & & \end{array}$$

(*Figures within parentheses show standard errors*)

(i) State the null hypothesis in terms of stationarity/nonstationarity. Give justification(s) for your choice.

(ii) Carry out the ADF test to conclude if the time series is stationary. (Note that the critical value of the ADF test statistic at 5% level of significance is -3.51.)

9. Suppose that you are testing $H_0 : p = \frac{1}{2}$ against $H_1 : p = \frac{2}{3}$ for a binomial variable X with $n = 3$. What values of X would you assign to the critical region if you wish to have $Prob$[Type I Error] $\leq \frac{1}{8}$ and you wish to minimize $Prob$[Type II Error]?

10. (a) Consider a linear regression model where the model for observation t is $y_t = \beta_0 + \beta_1 x_t + \varepsilon_t$. Assume $E(\varepsilon_t) = 0$ and $E(\varepsilon_t^2) = \sigma^2$, for all t, $E(\varepsilon_t \varepsilon_{t-s}) = E(\varepsilon_t \varepsilon_{t+s}) = 0$, for all $t \neq s$, and $E(x_t \varepsilon_t) = 0$, for all t.

(i) Suppose we do not observe y but instead observe $z = y + v$ where the error v is independent of x. If z is regressed on x, would the ordinary least squares estimate of the slope coefficient be a consistent estimator of β_1?

(ii) Suppose we do not observe x but instead observe $w = x + u$ where the error u is independent of x. If y is regressed on w, would the ordinary least squares estimate of the intercept be a consistent estimator of β_0?

(b) With reference to the model, $y_t = \mu + \varepsilon_t$, suppose we are given a sample of T observations. Assume, for all t, $E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = \sigma^2$, and $E(\varepsilon_t \varepsilon_{t-1}) = E(\varepsilon_t \varepsilon_{t+1}) = \rho\sigma^2$, and, for all t and $s > 1$, $E(\varepsilon_t \varepsilon_{t-s}) = E(\varepsilon_t \varepsilon_{t+s}) = 0$. Is the sample mean $(\frac{1}{T} \sum_{t=1}^{T} y_t)$ a consistent estimator of the population mean μ?