Sample Question 2019

TEST CODE: QEA

1. Answer the following questions.

   (a) Let $f : [0,1] \to [0,1]$ be a twice differentiable convex function. Define
   $$g(x) = [f(x)]^2 \ \forall \ x \in [0,1].$$
   Is g convex? Justify your answer. [4 marks]

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as
   $$f(x) = \frac{|x|}{2x} \ \forall \ x \in \mathbb{R} \setminus \{0\}.$$
   Can $f(0)$ be defined in a way such that f is continuous at 0? Justify your answer. [4 marks]

2. Let $f : [0,1] \to [0,1]$ be a strictly increasing function with $f(0) = 0$ and $f(1) = 1$. Define a function $g : [0,1] \to [0,1]$ such that $g(f(x)) = x$ for all $x \in [0,1]$.

   (a) Prove that $g'(y) = \frac{1}{f'(g(y))}$. [4 marks]

   (b) Is g strictly increasing? [2 marks]

   (c) If $f(x) = x^2$ for all $x \in [0,1]$, then find $g'(\frac{1}{2})$. [2 marks]

3. Three courses, A, B, and C are offered to a class of 100 students. The number of students in each course is: A is taken by 50 students, B by 80 students, and C by 40 students. Out of all the students, 20 have taken all the three courses. Every student has taken at least one course.

   (a) How many students have taken exactly two courses? [4 marks]

   (b) How many students have taken exactly one course? [4 marks]

4. Suppose A is any 2×2 matrix with a_{ij} denoting the entry of i-th row and j-th column. Suppose $a_{11}a_{22} = a_{12}a_{21}$.

   (a) Prove that one of the eigenvalues of A is zero. [4 marks]

   (b) Suppose A is the following matrix. Find all its eigenvalues. [4 marks]
   $$A = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$$

5. Consider the 3×3 matrix X shown below, where x, α are real numbers.

$$X = \begin{pmatrix} x & \alpha & 1 \\ x & x & 1 \\ 6 & 0 & 1 \end{pmatrix}$$

   (a) Show that if $x = \alpha$ then X does not have full rank. [2 marks]

   (b) Show that if $x = 6$ then X does not have full rank. [2 marks]

   (c) Show that if X does not have full rank, then either $x = 6$ or $x = \alpha$. [4 marks]

6. Answer the following questions.

   (a) Consider the following optimization problem:

$$\max_{x \in [0,\beta]} x(1-x),$$

   where $\beta \in [0,1]$. Let x^* be an optimal solution of the above optimization problem. For what values of β will we have $x^* = \beta$? [6 marks]

   (b) A firm is producing two products a and b. The market price (per unit) of a and b are respectively 3 and 2. The firm has resources to produce only 10 units of a and b together. Also, the quantity of a produced cannot exceed double the quantity of b produced. What is the revenue-maximizing production plan (i.e., how many units of a and b) of the firm? [6 marks]

7. Consider two random variables: (1) rain level (r), which can be either HIGH or LOW and (2) school attendance (s), which can be either 0 or 1. The joint probability distribution of (r, s) is as follows:

$$\text{Prob (HIGH,0)} = 0.2, \text{Prob (LOW,0)} = 0.3,$$

$$\text{Prob (HIGH,1)} = 0.35, \text{Prob (LOW,1)} = 0.15.$$

   (a) Find the marginal distribution of rain level. [4 marks]

   (b) Find the conditional probability of HIGH rain given that attendance is 0. [4 marks]

   (c) Are the two random variables independent? Justify your answer. [4 marks]

8. A slip of paper is given to person A, who marks it with either $(+)$ or $(-)$. The probability of her writing $(+)$ is $\frac{1}{3}$. Then, the slip is passed sequentially to B, C, and D. Each of them either changes the sign on the slip with probability $\frac{2}{3}$ or leaves it as it is with probability $\frac{1}{3}$.

(a) Compute the probability that the final sign is (+) if A wrote (+). [3 marks]

(b) Compute the probability that the final sign is (+) if A wrote ($-$). [3 marks]

(c) Compute the probability that A wrote (+) if the final sign is (+). [6 marks]

9. There are n houses on a street numbered $h_1, \ldots, h_n$. Each house can either be painted BLUE or RED.

   (a) How many ways can the houses $h_1, \ldots, h_n$ be painted? [2 marks]

   (b) Suppose $n \geq 4$ and the houses are situated on n points on a circle. There is an additional constraint on painting the houses: exactly two houses need to be painted BLUE and they cannot be next to each other. How many ways can the houses $h_1, \ldots, h_n$ be painted under this new constraint? [5 marks]

   (c) How will your answer to the previous question change if the houses are located on n points on a line. [5 marks]

10. There are n biased coins $C_1, C_2, \ldots, C_n$. They are tossed sequentially - coin C_1 is tossed first, followed by C_2, ..., finally C_n. The probability of heads in coin C_k for each $k \in \{1, \ldots, n\}$ is $\frac{1}{2k+1}$. For every k, let P_k be the probability that we have odd number of heads after coins $C_1, \ldots, C_k$ are tossed.

    (a) Find P_1, P_2, P_3. [3 marks]

    (b) For every $k \in \{1, \ldots, n\}$, write P_k as a function of P_{k-1}. [5 marks]

    (c) What is the value of P_n? [4 marks]

Sample Question 2019
TEST CODE: QEB

Group A

1. A judge is faced with two women, a and b, who both claim to be the mother of a child. The judge does not know the true mother. If the true mother gets the child, then she gets a utility of 100. On the other hand, the woman who is not the true mother only gets a utility of 50 if given the child. By not getting the child, both women get zero utility.

   The judge sets up the following game.

Step 1. He will ask a whether the child is hers. If she answers negatively, the child will be given to b. If she answers affirmatively, the judge will continue to the next step.

Step 2. He will ask b whether the child is hers. If she answers negatively, the child will be given to a. If she answers affirmatively, the judge will ask b to pay 75 and a to pay 10, and give the child to b. Utility from money is linear, i.e., paying p reduces utility by p.

   Since the judge does not know the true mother, there are two extensive form games induced - denote them as Γ_a (where a is the true mother) and Γ_b (where b is the true mother).

   (a) Describe Γ_a and Γ_b (using game tree representation). [6 marks]

   (b) Describe a strategy profile in Γ_a. [4 marks]

   (c) Describe all pure strategy Nash equilibria of Γ_a and Γ_b. [6 marks]

   (d) Argue that there is a unique subgame perfect equilibrium of each of the games where the true mother gets the child. Write down the subgame equilibrium strategy profiles in both the games. [9 marks]

2. Imagine a country consisting of several states. Within it the state of Devi Pradesh (DP) has the following farm production function. The aggregate farm output X is given by

$$X = AF(L_f, N_f),$$

   where A denotes the total factor productivity parameter, L_f is the agricultural labour input, and N_f is the agricultural land input. The production function is increasing in each of its inputs and is strictly concave. Let $\bar{N}$ be the total land in the economy.

The state also produces a non-farm output Z, produced using only labour and exhibits constant returns in the amount of labour employed, L_n:

$$Z = BL_n,$$

where B is the total factor productivity parameter.

There are 11.6 million consumers in the state of DP. Each of them is characterized by the same utility function, $u(x, y) = \alpha x^{0.3} y^{0.7}$, where x and y are quantities of farm and non-farm good consumed respectively by the consumer.

Now, suppose the state of DP is a small open economy, i.e., the prices of the outputs are assumed to be set at the national level (P_f and P_n) unaffected by the state's production. Assume labour to be immobile across states. Also assume that in equilibrium the states produces both goods and that labor is perfectly mobile across both sectors within a state. The productivity parameters and the prices are exogenously given.

(a) Write down the equilibrium condition for perfect labour mobility between sectors within the state. What does this condition solve for? [8 marks]

(b) Show (i) an increase in non-farm productivity decreases equilibrium farm employment and (ii) an increase in farm productivity increases equilibrium farm employment. [6 marks]

(c) Also argue that the equilibrium wage rate has zero elasticity with respect to farm productivity, and unitary elasticity with respect to non-farm productivity. [6 marks]

(d) Do these results depend on consumer preferences? Why not? [5 marks]

3. Suppose a monopoly firm produces two products, A and B, without any cost. Suppose every consumer wants at most one unit of good A, as well as one unit of good B. Consider a consumer such that her valuation of one unit of good A is v_A, and that for one unit of good B is v_B. If the products are bundled, then her valuation of the bundle containing one unit of good A and one unit of good B is $v_A + v_B$. Suppose (v_A, v_B) is uniformly distributed over the unit square.

(a) Solve for the optimal pricing of the monopolist when the goods must be sold separately. [8 marks]

(b) Solve for the optimal pricing of the monopolist when the goods must be sold as a bundle. [8 marks]

(c) Does the monopolist prefer bundling over selling separately? [4 marks]

(d) Does a welfare-maximizing firm prefer bundling over selling separately? [5 marks]

Group B

Be precise and brief in your answers. Wherever possible, use standard results and formulae (you do not have to show derivations) in addition to providing intuition.

1. An economist is asked to evaluate whether public expenditure on education *causes* an improvement in human capital. To test this hypothesis, she collects survey data on children in India over ten years (denote the years by $t = 1, 2, \ldots, 10$). The outcome of interest, Y, is daily hours of schooling for a child c living in a household h that resides in district d. The surveyed children are in the age group 6 to 14 and the sampling within a survey year and across survey years is independent. The survey datasets also contain information on individual characteristics of each child (denote the vector of these characteristics by I) and household level characteristics (H). These data are then matched with the district level public expenditure on education (E) for each year. The expenditures have been adjusted for inflation.

   Consider the empirical model

   $$Y_{chdt} = \beta_0 + \beta_1 E_{dt} + \epsilon_{chdt}$$

   Let β_1 be estimated by Ordinary Least Squares (OLS).

   (a) The economist now adds I and H to the empirical specification. What is the implication of these additions on the standard error of the estimated β_1? [5 marks]

   (b) Districts in India are typically very different from each other. Explain when this will affect the consistency of the estimated β_1? [2 marks]

   (c) Suppose she also includes district level dummy variables to the specification (in addition to I and H) and allows for an all India level trend. She uses OLS to estimate the specification but is told that the estimated β_1 is still inconsistent. When will this be the case? (In addition to stating the technical conditions, explain what they mean intuitively in the stated context) [5 marks]

(d) Suppose the economist is able to find a variable (z) that correlates with district level public expenditures over time but is not likely to be correlated with ϵ. Describe how this information can be used to estimate β_1 consistently. [8 marks]

(e) In part (d), we are now told that if one regresses E on z, I and H, the F statistic for the overall significance of the regression is 4. This is above the threshold value for the overall significance of regressors but not very high. Explain the consequences of this information for the results? [5 marks]

2. (a) Suppose you want to estimate the following population model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \lambda q + \epsilon$$

where x_1 and x_2 are not correlated with ϵ. You are interested, in particular, to get consistent estimates of β_1 and β_2. However data on q is missing. Instead, you have a proxy variable for q given by z, such that

$$q = \theta_0 + \theta_1 z + r$$

where $Cov(z, r) = 0$. You estimate the following empirical model using OLS:

$$y = \gamma_0 + \gamma_1 x_1 + \gamma_2 x_2 + \tau z + \omega$$

For each of the following circumstances, is $\hat{\gamma}_i$ a consistent estimator of β_i for $i = 1, 2$?

i. $Cov(x_k, r) = 0$ for all k but $E[r] \neq 0$ [5 marks]

ii. $E[r] = 0$ but $Cov(x_2, r) \neq 0$ [5 marks]

iii. $Cov(x_k, r) = 0$ for all k but $Cov(z, \omega) \neq 0$ [5 marks]

(b) You present the following regression results in a project presentation:

$$y = \underset{(2.06)}{0.54} + \underset{(2.53)}{3.23} x_1 - \underset{(0.72)}{0.25} x_2$$

where t statistics are reported below the estimated coefficients. Someone suggests dropping x_2 from the regression.

i. Elaborate on the pros and cons of dropping x_2. What will be your final verdict? [5 marks]

ii. Would you drop x_2 if y is log of (hourly) wage, x_1 is years of education, and x_2 is hours worked? Elaborate. [5 marks]

3. Answer the following questions precisely

(a) Consider the following time series:

$$y_t = \mu + \phi y_{t-1} + \epsilon_t$$

where ϵ_t is white noise.

i. If ϕ is 1, derive the mean and variance of the series. Is this series stationary? (provide a rationale for your answer) [5 marks]

ii. if ϕ is positive but strictly less than 1, show that the Auto-correlation Function (ACF) declines geometrically [5 marks]

(b) A government has instituted an employment guarantee scheme (EGS), beginning 2006, which aims to alleviate poverty. The program commits to providing employment at a fixed wage-rate for 100 days in a year to any household which demands employment. Access to the program is universal (anyone is eligible), participants must work to get paid and the wage rate is set at the minimum wage for the state (which varies across states). You want to evaluate the causal impact of this program on reducing poverty. The implementation of the program is staggered across time and districts. Districts are ranked in order of a backwardness index and the program is implemented, first, in the poorest 100 districts (in 2006) and then to rest of the districts (in 2008). The ranking of the districts in terms of the backwardness index are available to you.

i. Suppose you are told that you will be given access to a dataset with all relevant variables for *only* one year. There is no retrospective information in the dataset. Which year would you want that to be? Briefly identify and explain the main challenge in evaluating the causal impact of EGS on poverty alleviation with data for this one year. Can you suggest a reasonable strategy to evaluate the causal impact? State clearly the assumptions under which your strategy would be valid? (In case something is not specified in the question, you are free to make assumptions about the context and the dataset) [5 marks]

ii. Now suppose you also have access to a similar dataset collected in 2005. You have information for the same set of households. Write down and explain your empirical model to estimate the program's effect on poverty. State any assumptions you are making. And if possible, suggest ways to test these assumptions. [10 marks]

Group C

1. Consider a Solow growth model in continuous time with the following per-capita production function

$$f(k) = k^4 - 6k^3 + 11k^2 - 6k.$$

Here $k = \frac{K}{L}$, where K is the aggregate capital stock and L is labor supply. Labor grows at the rate $n > 0$ and there is no growth in technological progress. Depreciation rate on capital is $\delta \in [0, 1]$ and exogenous saving rate is $s \in [0, 1]$.

   (a) Verify which of the Inada conditions with respect to K and L are violated. [8 marks]

   (b) Show that with this production function, there exist three steady state equilibria. Which of these are locally stable/unstable? Plot your answer. [12 marks]

   (c) Explain your answer to part (b). [5 marks]

2. Consider an Overlapping Generations (OLG) model in which there is a constant population of identical two period lived generations with utility functions given by

$$u_t = log(c_t) + \beta log(c_{t+1}),$$

where c represents consumption and $\beta > 0$. Each generation is supplied with one unit of labor in youth, no labor supply in old age, and a Leontief technology,

$$Y_t = A \min(K_t, L_t).$$

where Y_t, K_t, L_t represent output, aggregate capital, and aggregate labor respectively. A is a positive constant. In the first period, the consumer budget constraint is given by $c_t + s_t = w_t$ where s_t is the saving and w_t is the wage. In the second period, the constraint is given by $c_{t+1} = s_t R_{t+1}$ where R_{t+1} is the gross rate of return on capital in $t + 1$.

   (a) Write down the expression for the wage rate and the rental rate as functions of $k = \frac{K}{L}$. Assume that capital depreciates at a rate $\delta = 1$. [5 marks]

   (b) Find all possible steady states, and describe their stability. [10 marks]

(c) Discuss the dynamics of the economy by plotting the law of motion for per-capita capital. [10 marks]

3. Consider an economy where agents are identical and they live for three periods. Suppose in the first period they invest e in their education and become skilled in the second period where the level of skill $h = e^{\delta}$, $0 < \delta < 1$. The investment in education is done via borrowing from the market at a fixed rate R, that is, per unit of borrowing costs $R > 1$ per period. In the second period of their life they work using the acquired skill (h). The total wage of an agent with skill level h is given by wh where w is the exogenously given wage rate. Once they earn wage income wh in the second period, they repay their total borrowing for education. Assume that the wage income is sufficient to repay this education loan. Further, for simplicity we assume that they do not consume anything in the first period. In the second period they take their consumption and saving decision for the second and the third period. Gross return on saving is the same R per unit per period. Agents do not work in the third period and live on their saving made in the second period. Suppose the life-time utility function is given by

$$u = u(c_t) + \beta u(c_{t+1})$$

where c_t and c_{t+1} are the consumption levels in the second and third periods respectively. The function u is assumed to be strictly positive and strictly concave with $\beta > 0$.

(a) Write down the utility maximization problem of the agents. Clearly derive the first order conditions with respect to saving (s) and investment in education (e) in this maximization problem. [10 marks]

(b) Derive the optimal level of investment in education (e). Describe its relationship with the wage rate (w) and the rate of interest (R). [5 marks]

(c) Suppose agents get utility from consumption that is over and above $\alpha\bar{c}_t$ and $\alpha\bar{c}_{t+1}$ in the second and third period respectively. $\bar{c}_t$ and $\bar{c}_{t+1}$ are the average consumption levels of the society in the second and third period respectively, $0 < \alpha < 1$, and they are assumed to be given to the agents. That means the revised utility function is given by

$$u = u(c_t - \alpha\bar{c}_t) + \beta u(c_{t+1} - \alpha\bar{c}_{t+1}).$$

Derive the first order conditions of the revised utility maximization problem with respect to saving (s) and investment in education (e) .

Solve for the modified level of investment in education. Discuss your answer. [10 marks]