QEA 2019

1. Consider the set, A, of all rectangles of area 100 square units. Let $a \in A$ be a rectangle which minimizes the sum of the width and twice the length among all members of A. Find the length and the width of every such a.

2. Prove or disprove the following statement: the function $x \ln_e x$ defined on the positive orthant $\mathbb{R}_{++}$ is convex.

3. Consider a non-negative random variable X with the distribution function F. Suppose that F is strictly increasing and differentiable. Prove or disprove the following statement: the expected value of X can be represented as
$$\int_{t=0}^{t=1} F^{-1}(t)dt.$$

4. A class has 100 students. Let a_i, $1 \leq i \leq 100$, denote the number of friends the i-th student has in the class. For each $0 \leq j \leq 99$, let c_j be the number of students having at least j friends. Derive the value of $\sum_{i=1}^{100} a_i - \sum_{j=1}^{99} c_j$.

5. Consider the 3×3 matrix
$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$
Find all the eigenvalues of A, A^2, and $A + 5I$, where I is the identity matrix of order 3×3.

6. Suppose that the probability of getting heads is 0.4 for Coin 1 and the probability of getting heads is 0.7 for Coin 2. One of these coins is randomly chosen and then the chosen coin is flipped 10 times.
   (a) What is the probability of getting exactly 7 heads out of the 10 flips?
   (b) Given that the first of these ten flips is heads, what is the conditional probability that exactly 7 out of the 10 flips gives heads?

7. Suppose that the equation $f(x) = 2x^3 + 3x^2 - 11x - 6 = 0$ has at least one integer root. Find all the roots of $f(x)$ by using this information.

8. There are w white balls and b black balls in a box. Balls are drawn one after another without replacement. What is the probability that the white balls will be exhausted before the black balls?

9. Let $f : [0,1] \to [0,1]$ be a strictly convex and continuous function such that $f(0) = 0$ and $f(1) = 1$. Consider a point $x \in (0,1)$. Let A, B, C, and D be the points $(0,0), (1/2, 1/2)$, $(1,1)$, and $(x, f(x))$, respectively. What is the ratio between the areas of the triangles $\triangle ABD$ and $\triangle BCD$?

10. Suppose $x^2 + 1 \geq (1 + \gamma)x$ is true for all $x \in \mathbb{R}$ if and only if $\gamma \in [a, b]$. Find the values of a and b. Here, $\mathbb{R}$ is the set of all real numbers.

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Group A

1. Suppose an individual's decision problem is to make choices from a two good set X and assume that $X = \mathbb{R}^2_+$, where $\mathbb{R}_+$ is the set of non-negative real numbers. The binary relation 'at least as good as' which is denoted by $\succeq$ on X allows for comparing pairs of alternatives $x, y \in X$. Here $x \succeq y$ means that x is 'at least as good as' y. The preference relation $\succeq$ on X is rational if it is complete (that is, for all $x, y \in X$, either $x \succeq y$ or $y \succeq x$ or both) and transitive (that is, for all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$). A preference relation $\succeq$ on X is monotone if for any $x = (x_1, x_2) \in X$ and any $y = (y_1, y_2)$ such that $y_i > x_i$ for all $i = 1, 2$, we have $y \succ x$. A preference relation $\succeq$ on X is locally non-satiated if for any $x \in X$ and any $\epsilon > 0$, there exists $y \in N_\epsilon(x) \cap X$ such that $y \succ x$. Here, by $N_\epsilon(x)$ we denote the ϵ-neighborhood around x. A preference relation $\succeq$ on X is continuous if for any two sequences (x_n) and (y_n) in X such that (x_n) converges to x, (y_n) converges to y, and $x_n \succeq y_n$ for all $n \in \mathbb{N}$, we have $x \succeq y$. Here, $\mathbb{N}$ denotes the set of all natural numbers. A utility function $u : X \to \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, assigns a numerical value to each element in X. A utility function $u : X \to \mathbb{R}$ represents a preference relation $\succeq$ on X if, for all $x, y \in X$, $x \succeq y$ if and only if $u(x) \geq u(y)$. Using the above information, answer the following questions.

   (a) Show that if the preference relation $\succeq$ on X is complete and satisfies monotonicity, then it also satisfies local non-satiation. **(8)**

   (b) Show that if the preference relation $\succeq$ on X has a utility representation, then $\succeq$ on X is rational. **(7)**

   (c) Show that if the utility function $u : X \to \mathbb{R}$ representing $\succeq$ on X is continuous, then the underlying preference relation $\succeq$ on X is continuous. **(10)**

2. Three students are arguing over a cake in a game theory class. The teacher, in a bid to teach them some game theory, decides the following bargaining scheme. Student 1 must propose a division (for three of them) of the cake. If Student 2 and 3 agree, then the proposed division is implemented. Otherwise, the teacher eats p proportion of the cake and gets Student 1 out of the class. Next, Student 2 must propose a division (for Student 2 and Student 3) of the remaining cake. If Student 3 agrees, then the proposal is implemented. Otherwise, the teacher eats q proportion of the (remaining) cake and gives the rest to Student 3 who eats it.

   Assume that the students always agree when they are indifferent and there is no discounting.

(a) Model this situation as an extensive form game. **(5)**

(b) Find all Nash equilibria of this game when $p = q = \frac{1}{2}$. What happens for other values of p and q? **(10)**

(c) Find all subgame perfect Nash equilibria of this game when $p = q = \frac{1}{2}$. Do there always exist subgame perfect Nash equilibria for arbitrary values of p and q? For what values of p and q will there be only one subgame perfect Nash equilibrium? **(10)**

3. Consider a simultaneous move two-player game, where each player announces an integer between 2 and 100 (both inclusive). If both players announce the same number x, then each of them receives rupees x. Otherwise, if they announce different numbers, say x and y, with $x > y$, then the player announcing x receives rupees $y - k$ and the player announcing y receives rupees $y + k$.

   (a) Model this game as a normal form game $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where N is the set of players, for all $i \in N$, S_i and u_i are the strategy set and pay-off function of player i, respectively. **(5)**

   (b) Find the best response function of each player for $k = 2$. **(7)**

   (c) Find all Nash equilibria of this game for $k = 2$. **(5)**

   (d) How will your answers to (b) and (c) change if $k = 1$? **(8)**

Group B

1. Consider an economy consisting of a large number of identical individuals. At the beginning of any period t, a representative individual has a capital stock K_t from his past accumulation which is used to produce an output y_t through the production function $y_t = \epsilon_t K_t$. Capital stock K_t depreciates completely after production. So output y_t is partly saved and invested to build up next period capital stock K_{t+1}. The remaining output is consumed and is denoted by C_t. An agent has a period utility $\ln C_t$ and a discount factor $\beta < 1$. He maximizes the discounted sum of his life time utility over an infinite horizon subject to his budget constraint each period. His choice variables at any period t are C_t and K_{t+1}. Suppose $\epsilon_t = \bar{\epsilon}$ for all t.

   (a) Write down the maximization problem of the representative agent. **(5)**

   (b) Write down the first order conditions of maximization. **(5)**

   (c) In a steady state, consumption, capital stock and income grow at the same constant rate over time. Find the steady state rate of growth. Also find the fraction of income that is consumed and the fraction of income that is saved in steady state. **(10)**

   (d) Now suppose there is a temporary productivity shock in period t (the current period) increasing ϵ_t to $\tilde{\epsilon} > \bar{\epsilon}$. For all other periods, the value of ϵ remains pegged to $\bar{\epsilon}$. Find the effects of this temporary increase in productivity on consumption (current and future) and capital stock (current and future). **(5)**

2. Consider a Solow-style economy that produces a homogenous output using two types of capital, private (K_t) and public (G_t) according to the technology

$$Y_t = K_t^{\alpha} G_t^{\beta} \tag{1}$$

   where $\alpha + \beta < 1$. Both types of capital depreciate completely upon use. In each period, the government taxes income at the rate $\tau \in [0, 1]$ and invests the proceeds in public capital in the next period. Agents save a fixed fraction, $s \in [0, 1]$ of their after tax income and invest it in private capital. Hence,

$$K_{t+1} = s(1 - \tau)Y_t \tag{2}$$

   and

$$G_{t+1} = \tau Y_t. \tag{3}$$

(a) Using (1)-(2)-(3), derive a single equation that describes the evolution of income, i.e., an equation that describes Y_{t+1} in terms of Y_t. **(5)**

(b) Solve for the steady state value of Y and show that the system is stable. **(5)**

(c) How should the steady state value of Y vary with s and τ? Discuss your answer intuitively. **(5)**

(d) What value of τ should the government choose if it wants to maximize steady state output? Discuss your answer intuitively. Also, plot a graph to explain your answer. **(10)**

3. Consider an economy producing a single final good by means of labor. Technology is such that one unit of labor is required to produce one unit of the good. Furthermore, there is zero profit in production of the good. Let P denote the price of the good.

   Agents have a utility function given by

$$U = C^{3/4}(M'/P)^{1/4},$$

   where C is the consumption of the above mentioned good and M' is the end of the period holding of money. Money is the only asset and agents are endowed with $M = 100$ units of money.

   Total labor endowment of the economy is given by $L = 300$.

   (a) Consider the case where price is completely flexible. Under such circumstance, solve for the equilibrium level of price. Further, show what will happen to price and output if M increases from 100 to 200. **(15)**

   (b) Say now for some reason price is rigid and is fixed at $P = 5$. Solve for the equilibrium level of output. What would happen to output in this alternative scenario if M increases from 100 to 200? **(10)**

Group C

1. Consider the linear regression model

$$y_i = \alpha + \beta x_i + e_i, \;\; i = 1, 2, \ldots, n,$$

where $X = (x_1, x_2, \ldots, x_n)'$ is non-stochastic with finite second moment and $\{e_i\}$ is independently distributed random variable with mean zero and variances σ_i^2. Consider the quantity $S^2 = (Y - \underset{\sim}{\alpha} - \beta X)' W (Y - \underset{\sim}{\alpha} - \beta X)$, where $Y = (y_1, y_2, \ldots, y_n)'$, $\underset{\sim}{\alpha} = (\alpha, \alpha, \ldots, \alpha)'$, and W is a $n \times n$ known symmetric matrix. Let Σ be the variance covariance matrix of $\underset{\sim}{e} = (e_1, e_2, \ldots, e_n)'$. Let $\theta = (\alpha, \beta)'$.

   (a) Let $W = XX'$. Minimize S^2 with respect to θ. Show that this minimizer is an unbiased estimate of θ. **(5)**

   (b) Let $W = \Sigma^{-1}$. Minimize S^2 with respect to θ. Show that this minimizer is an unbiased estimate of θ. **(5)**

   (c) Compare the relative efficiencies of the above two estimators, *viz.* the estimator from (a), and the estimator from (b), respectively. **(7)**

   (d) Suppose you want to test the null hypothesis, $H_0 : \beta = 0$. How would you test the hypothesis using the estimator you obtained in (a)? **(8)**

2. (a) Let $\{X_t, t = 1, 2, \ldots\}$ be a sequence of random variables with mean zero and unit variance. Assume that $X_0 = 0$. Define $Z_t = X_t X_{t-1}$ for all $t = 1, 2, \ldots$. Examine whether Z_t is weakly stationary or not for the following cases:
   (i) X_t is white noise. **(5)**
   (ii) X_t is i.i.d. process. **(5)**

   (b) Let $\{X_t, t = 1, 2, \ldots\}$ be a sequence of independent normal random variables with mean zero and unit variance. Consider the following process:

$$W_t = 10 + 1.5 W_{t-1} - 0.56 W_{t-2} + X_t,$$

   where $W_0 = W_{-1} = 0$.
   (i) Show that the process is stationary. **(5)**
   (ii) Find the mean and variance of the process W_t. **(3+7)**

3. A researcher wants to estimate the earnings returns to studying Science in higher secondary school. She surveys men in the age group

25-60 (the average age in the sample is 40) with two pre-conditions: that they have completed class XII and that they are working. To begin with, assume everyone is working (no unemployment) and that it is possible to calculate each person's earnings. Let **Dummy: Science** take the value 1 if the person studied science in higher secondary school, 0 if the person studied commerce or arts. **Dummy 1st Division** indicates the person got 1st division (60 % and above) in class X exams; **Dummy 2nd Division** indicates that the person got second division (50 % to less than 60 %) with the reference category indicating that the person received marks below 2nd division. In addition, there are controls for english language fluency with no fluency being the reference category. **Age** refers to the age of the person (a proxy for work experience). The reference category for caste is every caste except those that are notified as Scheduled castes. **HH Education** refers to the average education of household members in the family. **Max Parent Education** refers to the education of the most educated parent. Since we are looking at individuals who are between 25 and 60, many parents don't live with the individuals or are dead and we do not have the education level for these parents. So the sample for this regression (given in column 5) is based on smaller number of observations. Each column indicates a separate OLS regression. In columns (3)-(5), she controls for district level dummies (dummy variables indicating district of residence). Regressions in columns (1) and (2) are estimated without district dummies.

(a) Results in column (4) are reported by the researcher as the main results. Write a short note explaining the results. **(4)**

(b) Looking at the coefficient of **Dummy: Science** across specifications (1)-(4) can you comment on what the most important control variables are, when you want to estimate the returns to studying science? Give some intuition for your answer. **(2)**

(c) Compare the standard error of the coefficient of **Dummy: Science** in columns (1) and (2). What has contributed to fall in the standard error in column (2)? Is it possible to say before running the regression in (2) that the standard error would fall? **(2+2)**

(d) In the specification in column (5), addition of a new variable necessitates dropping sample size. In theory, what should be the impact of the drop in sample size on your results? (Hint: focus on the coefficient and standard error of **Dummy: Science**). **(4)**

(e) Why do you think the R-squared value more than doubles from the specification in column (2) to column (3)? **(2)**

(f) You are critically evaluating these regression results. As pointed out above, the main specification is column (4). Given all the results (all columns), what sort of concerns should you have about the results? In other words, if you wanted the author to collect data on more variables, what sort of additional data would you ask the author to collect to add to this specification? **(4)**

(g) An evaluator of the research paper tells the researcher that collecting data on more variables is not sufficient to provide a consistent estimator of the coefficient of **Dummy: Science**. A Randomised Control Trial is not feasible. You are thinking of suggesting an Instrumental Variables estimator. What are the considerations when you are thinking about specifying the kind of variables that can serve as instruments? Can the economic status when the individual was in Class XII be used as an instrument? **(3+2)**

Table 1: Returns to Science Major

Dependent Variable:	Log(Earnings)				
	(1)	(2)	(3)	(4)	(5)
Dummy: Science	0.36***	0.20***	0.25***	0.22***	0.21***
	(0.06)	(0.05)	(0.04)	(0.04)	(0.05)
Dummy: 1st Division		0.33***	0.22***	0.21***	0.18*
		(0.08)	(0.07)	(0.07)	(0.10)
Dummy: 2nd Division		0.08	0.02	0.02	0.00
		(0.07)	(0.06)	(0.06)	(0.08)
Dummy: Moderately Fluent English		0.08	0.14**	0.11**	0.08
		(0.09)	(0.05)	(0.04)	(0.05)
Dummy: Extremely Fluent English		0.41***	0.42***	0.35***	0.36***
		(0.11)	(0.06)	(0.05)	(0.06)
Age				0.06***	0.03
				(0.02)	(0.02)
Age Square				-0.002***	-0.001
				(0.0005)	(0.001)
Dummy: Scheduled Castes				-0.06	-0.06
				(0.045)	(0.07)
HH Education				0.03***	0.03***
				(0.00)	(0.01)
Max Parent Education					0.01
					(0.01)
Constant	4.65***	4.39***	4.41***	2.58***	3.07***
	(0.06)	(0.13)	(0.08)	(0.30)	(0.32)
District Dummies	No	No	Yes	Yes	Yes
Observations	4,763	4,763	4,763	4,687	2,513
R-squared	0.03	0.10	0.25	0.30	0.36

NOTES: Robust standard errors in parentheses. *** p<0.01, ** p<0.05, * p<0.1. There are 723 districts in India.