1. Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined as follows: $g(x) = f(x) + f(2+x)$, where $f(x) = 1 - |x|$ if $|x| \leq 1$ and $f(x) = 0$ if $|x| > 1$.

   Is the function g differentiable at every point in $\mathbb{R}$? If not, at which points is it not differentiable? **(10)**

2. Consider the following system of equations:

$$\begin{aligned} x - y + 3z &= c \\ x + 3y - 3z &= 2c \\ 5x + 3y + 3z &= c^2, \end{aligned}$$

   where c is some real number.

   (a) Does this system have a unique solution for some values of c? If yes, then find those values of c. **(5)**

   (b) Is this system consistent for some values of c? If yes, then find those values of c. **(5)**

3. (a) It is decided that 10 students will be selected for the interviews based on the JRF QE written test scores. For interviewing purpose, these students are to be *partitioned* into three groups each having at least 2 and at most 4 students. In how many ways can this be done? **(5)**

   (b) Suppose that exactly 5 girls and 5 boys are selected for the interviews. As before, these students are to be partitioned into three groups each having at least 2 and at most 4 students, but now with the additional criteria that there must be at least one girl and at least one boy in each group. In how many ways can this be done? **(5)**

4. Let x_1, x_2, x_3 be three roots of the equation $4x^3 + 3x^2 + 2x + 1 = 0$. Find the value of $x_1^2 + x_2^2 + x_3^2$. **(10)**

5. (a) Consider all triangles such that the length of one side is 1 unit and the area is 1 square unit. Which one of these will have minimum perimeter? In other words, suppose T is the set of all triangles $\triangle ABC$ such that the length of the side AB is 1 unit and the area of $\triangle ABC$ is 1 square unit. Find the length of the sides of a triangle $\triangle XYZ$ in T such that $p(\triangle XYZ) \leq p(\triangle PQR)$ for all $\triangle PQR \in T$. Here, $p(\triangle PQR)$ denotes the perimeter of the triangle $\triangle PQR$. **(5)**

   (b) Consider all triangles having 1 square unit area. Which one of these will have the minimum perimeter? **(5)**

6. State whether the following statements are true or false. If true, provide a proof, and if false, provide a counter example.

(a) Let f and g be two non-negative convex functions on $[0, 10]$. Then, the function $h : [0, 10] \to \mathbb{R}$, defined as $h(x) = f(x)g(x)$ for all $x \in [0, 10]$, is a convex function. **(5)**

(b) If $f : \mathbb{R} \to \mathbb{R}$ is a convex function, then f^2 is also convex. **(5)**

7. A student, E, has missed his Statistics exam. He requests his Statistics teacher, T, to give him some grace marks. T agrees to give him some random score and proposes the following scheme: "I will chose a random number (that is, with uniform probability) from a set S of scores. If the number is bigger than 45, you will get 45. Otherwise, you will get the same score as the chosen random number." When E agrees to this proposal, T asks him to chose one of following two options for S: (i) $S = \{0, 1, 2, \ldots, 100\}$, that is, the set of all integers between 0 and 100, and (ii) $S = [0, 100]$, that is, the set of all real numbers between 0 and 100. E considers both his expected score and the variance of his score to decide an option: (i) if an option gives him higher expected score he will prefer that (regardless of the variance), (ii) if both the options give him the same expected score but one has lower variance than the other, he will prefer the one with lower variance, and (iii) if both the options have the same expectation and variance, he will be indifferent over those.

   What should E's preference be over the two options and why? **(10)**

8. What is the total number of rectangles in a chess board? What is the total number squares in a chess board? (There are 8 rows and 8 columns in a chess board.) **(10)**

9. Consider the matrix $A_{n\times n}$ such that $a_{k,k-1} = a_{k,k+1} = k$ for all $k = 2, 3, \ldots, n-1$, $a_{1,2} = 1$, $a_{n,n-1} = n$, and $a_{i,j} = 0$ for all other cases. What is the rank of $A_{n\times n}$? **(10)**

10. Suppose $\{x_1, x_2, \ldots, x_n\}$ is a set of n positive numbers, where $n \geq 2$. Let the arithmetic mean and the standard deviation be

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}, \text{ respectively.}$$

   Show that the coefficient of variation, defined to be the ratio between standard deviation and mean, that is $s/\bar{x}$, is always strictly less than $\sqrt{n-1}$.
   (**Hint.** You may use the inequality $\sum_{i=1}^{n} x_i^2 < (\sum_{i=1}^{n} x_i)^2$.) **(10)**

- There are **9** questions in all: **3** questions in Group A, **3** questions in Group B, and **3** questions in Group C.
- Answer **4** questions in total taking **at least 1** question from each Group.
- Each question carries **25** marks.

Group A

1. Suppose an individual's decision problem is to make choices from a two good set X and assume that $X = \mathbb{R}_+^2$, where $\mathbb{R}_+$ is the set of non-negative real numbers. The binary relation 'at least as good as' which is denoted by $\succeq$ on X allows for comparing pairs of alternatives $x, y \in X$. Here $x \succeq y$ means that x is 'at least as good as' y. The preference relation $\succeq$ on X is rational if it is complete (that is, for all $x, y \in X$, either $x \succeq y$ or $y \succeq x$ or both) and transitive (that is, for all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$). A preference relation $\succeq$ on X is *monotone* if for any $x = (x_1, x_2) \in X$ and any $y = (y_1, y_2)$ such that $y_i > x_i$ for all $i = 1, 2$, we have $y \succ x$. Here $\succ$ on X represents the strict preference relation, that is, for any $x, y \in X$, $x \succ y$ means that $x \succeq y$ is true and $y \succeq x$ is not true. The preference relation $\succeq$ on X is *convex* if for any $x = (x_1, x_2) \in X$, any $y = (y_1, y_2) \in X$ and any $z = (z_1, z_2) \in X$ such that $y \succeq x$ and $z \succeq x$, we have $(\alpha y + (1 - \alpha)z) \succeq x$ for any $\alpha \in [0, 1]$. A utility function $u : X \to \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, assigns a numerical value to each element in X. A utility function $u : X \to \mathbb{R}$ represents a preference relation $\succeq$ on X if, for all $x, y \in X$, $x \succeq y$ if and only if $u(x) \geq u(y)$. Let $\succeq$ be a preference relation defined on X and let $u(.)$ be a utility function representing it. Using the above information, answer the following questions.

   (a) When do we say that the utility function $u(.)$ representing the preference relation $\succeq$ on X is increasing? **(3)**

   (b) Show that the preference relation $\succeq$ on X is monotone if and only if the utility function $u(.)$ representing it is increasing. **(2+2=4)**

   (c) When do we say that the utility function $u(.)$ representing the preference relation $\succeq$ on X is quasi-concave? **(4)**

   (d) Show that the preference relation $\succeq$ on X is convex if and only if utility function $u(.)$ representing it is quasi-concave. **(7+7=14)**

2. (No proof is required, but you need to explain your answer.)

   Suppose there are $n \geq 2$ firms producing a homogeneous good. Firm i has cost function $C_i = c_i q_i$ where c_i is a positive constant number. The market demand for the product is $P(\sum_i q_i)$ with $P' < 0$ for $\sum_i q_i > 0$. There is no capacity constraint. Firms are assumed to choose their prices. In case of a price tie, they share the market equally.

(a) Suppose $n = 3$ and $c_i = c$ for all i. Find all Nash equilibria. **(4)**

(b) Suppose $n = 2$ and $c_1 < c_2$. Also assume that the smallest legal tender is e, where $0 < e < c_1$. Find the equilibrium when the firms choose prices simultaneously and non-cooperatively. **(5)**

(c) Suppose $n = 2$ and $c_1 < c_2$ but medium of exchange is continuous (i.e., $e = 0$). Firms choose their prices sequentially. Find equilibrium prices when (i) firm 1 is choosing first, and (ii) firm 2 is choosing first. **(5)**

(d) Suppose $n = 2$, and the cost function is $C_i = F + cq_i, i = 1, 2$, with $F > 0$ if $q_i > 0$, and $F = 0$ if $q_i = 0$. Firms choose their prices simultaneously and non-cooperatively. What can you say about equilibrium prices? Will your answer change if, in case of a price tie, instead of sharing the market demand, one firm is selected randomly to serve the whole market? **(6)**

(e) Suppose $n = 2$ and $c_1 = c_2$ but the firms will have to decide first whether to enter the market and pay a fixed cost $F > 0$ or stay out. Then price competition follows. What can you say about equilibrium? **(5)**

3. Consider a household endowed with $\bar{l}$ amount of labour, which it has to allocate between urban employment and rural employment, so as to maximize expected utility. The household's indirect utility function is given by:

$$Eu(m) = \bar{m} - \frac{k\text{Var}_m}{2}; k > 0,$$

where $\bar{m}$ is mean household income and Var_m is the variance of household income.

(a) Show how the extent of urban employment changes with (i) labour endowment, and (ii) variance and covariance of rural and urban wage rates. **(20)**

(b) Show how the proportion of household labour allocated to urban employment changes as the household's total labour endowment changes. **(5)**

Group B

1. Consider a two sector dynamic model where capital accumulation satisfies the following equations of motion:

$$\dot{k_1} = \sqrt{k_1} - \theta k_1 \text{ with } 0 < \theta < 1, \text{ and}$$

$$\dot{k_2} = \sqrt{k_1 k_2} - \beta \text{ with } \beta > 0.$$

Here, $k_1 \geq 0$ and $k_2 \geq 0$ are capital stocks of the two sectors.

   (a) Find out the steady state equilibrium. **(5)**

   (b) Draw the phase diagram and show that this equilibrium point exists and is unique. **(10)**

   (c) Examine the nature of stability of this equilibrium. **(10)**

2. Consider a static economy consisting of two sectors – a consumption goods producing sector and an investment goods producing sector. Consumers consume a constant fraction of their money income on consumption. The rest is saved. Investment demand by firms is autonomously given and is fixed in real terms. In each sector there is a production function which uses labour and capital to produce final output. The production functions exhibit constant returns to scale and diminishing returns with respect to the inputs. In each sector, capital is fixed and immobile. The economy wide money wage rate is given and at that money wage rate some labour remains unemployed.

   (a) Set up the model and show how equilibrium income and employment are determined. **(10)**

   (b) Find the effect of a change in the money wage rate on income and employment. **(5)**

   (c) Find the effect of a technical progress (increasing marginal productivity of labour at each level of employment) in (i) the consumption goods sector, and (ii) investment goods sector. **(5+5)**

3. Following is a description of a closed economy.

   - **Consumption goods sector**
     There are two consumption goods produced in the economy. $Q_i^C, i = 1, 2$ denotes the production of the i-th consumption good. Q_i^C is produced by capital alone that is devoted to the consumption goods sector and is denoted by K^C. A constant $a_{K^C i}$ denotes the amount of capital required to produce one unit of the i-th consumption good.

Preferences are described by the utility function defined as: $U = (X_1^C)^\alpha (X_2^C)^{1-\alpha}$, where $0 < \alpha < 1$ and $X_i^C, i = 1, 2$ denotes the economy-wide aggregate consumption of the i-th consumption good.

- **Investment goods sector**
  Investment good I is produced by two intermediate goods in accordance to the production function: $I = (Q_1^I)^\gamma (Q_2^I)^{1-\gamma}$, where $0 < \gamma < 1$ and $Q_i^I, i = 1, 2$ denotes the amount of the i-th intermediate good used in the production of I. Q_i^I is produced by capital alone that is devoted to the Investment goods sector and is denoted by K^I. A constant $a_{K^I i}$ denotes the amount of capital required to produce one unit of the i-th intermediate good.

- **Allocation of I between sectors**
  A constant fraction θ of I, each time period t, is invested in the investment goods sector, and $(1 - \theta)$ of I is invested in the consumption goods sector to augment the capital stock K^I and K^C, respectively, i.e. $\frac{dK^I(t)}{dt} = \theta I(t)$ and $\frac{dK^C(t)}{dt} = (1 - \theta) I(t)$, where $z(t)$ denotes the value of the variable z at time t, and $\frac{dZ(t)}{dt}$ the time derivative of $z(t)$. Let $K^I(0)$ and $K^C(0)$ denote the initial stock of capital installed in the investment goods sector and consumption goods sector respectively.

Assume, all markets are competitive and all prices flexible.

(a) Find out the time paths of K^C, K^I, U and I. **(20)**

(b) Find out the long run growth rate of K^C, K^I, U and I (By long run is meant, t going to infinity.). **(5)**

Group C

1. Consider the time series

$$\begin{aligned} y_t &= \mu + x_t + z_t, \\ x_t &= \epsilon_t + \theta \epsilon_{t-1}, \text{ and} \\ z_t &= \beta z_{t-1} + u_t, \end{aligned}$$

where $\epsilon_t \overset{iid}{\sim} N(0, \sigma_\epsilon^2)$, $u_t \overset{iid}{\sim} N(0, \sigma_u^2)$, $|\beta| < 1$, and ϵ_t's and u_t's are independent of each other.

   (a) Show that y_t is an ARMA$(1, 2)$ process. **(8)**

   (b) Show that y_t is stationary. **(10)**

   (c) Derive the invertibility condition. **(7)**

2. A researcher wants to estimate the impact of caste diversity (CC) in a school on the average scholastic performance of children in the school. The focus is on schools in rural India that impart primary education. The typical source for such data in India is the District Information System for Education (DISE). DISE contains school infrastructure characteristics (number of classrooms, computer facilities, electrification status, etc), enrolment of students (for each class) and the caste composition of those enrolled (Scheduled Caste, Scheduled Tribes, Other Backward Classes, Others). It also contains data on how many students appeared for the class 5 examination in a school and how many such students got 60% and above. The village location of the schools is reported in the data. The schools have a unique identification number that does not change over the years. Suppose someone also provides you data on the village development index (sourced from census 2011) for the villages. Answer the following questions. If you feel additional assumptions need to be made, state them clearly.

   (a) Suppose you have access to data only for 2016.

      i. Write down an empirical model that can be estimated to obtain the parameter of interest. Clearly define the variables involved. In particular explain how you would define the outcome variable and caste diversity? **(2)**

      ii. The researcher finds that there are always at-least 2 schools in a village. What difference will knowing this fact make to the formulation of the empirical model? Explain intuitively what problem does this fact help her avoid? **(3)**

iii. Even after incorporating changes prompted by the presence of multiple schools in a village, some one points out that an Ordinary Least Squares (OLS) regression exercise may provide an incorrect estimate of the parameter. Provide what technical condition may be violated and explain the reason for the violation intuitively. **(2)**

(b) Suppose you had data on two years: 2014 and 2016.

i. Write down an empirical model that incorporates the panel nature of the data. **(2)**

ii. Suppose you estimate a fixed effects estimator using the empirical model you have written in the previous question. Explain how you expect it to perform on grounds of consistency and efficiency? **(4)**

iii. Assume that the fixed effects estimator is consistent. Suppose for each school i, you calculate the first difference of the dependent variable and each of the independent variables (A first difference is defined as $\Delta V \equiv V_{2016} - V_{2014}$) and are planning to run an OLS regression of the following empirical model:

$$\Delta Y_i = \beta_0 + \beta_1 \Delta CC_i + \Gamma \Delta Z_i + \psi_i \tag{1}$$

where Z is a vector of other school characteristics. You observe that the variance of ΔY is different for different values of the variable ΔZ^j ($Z^j \in Z$). You ignore this observation and estimate the equation above by OLS assuming that all the Classical Linear Regression assumptions go through.

A. Does it indicate that we should run some diagnostics tests after the OLS regression? What problem could such a diagnostic check indicate? **(3)**

B. Suppose the diagnostic test gives evidence that there is a problem. What implication will this problem have on the consistency and unbiasedness of the OLS estimator that has been used above? What can we say about the standard error of the estimated parameters? **(5)**

C. Suppose you are now allowed to re-estimate your model. What estimation procedure will you follow? **(4)**

3. The Maternity Benefit Act 2017 (Amendment) increased maternity leave for working women from 12 weeks to 26 weeks. This was enacted all over India starting 1st March 2017. You have been commissioned by the health ministry to evaluate the impact of the scheme on the health of

the children who are greater than 1 week old but less than 1 year old. You have access to survey data on children (indexed by i) in the years 2016 and in 2019 (indexed by t that takes the values 0 and 1 for the years 2016 and 2019 respectively). The surveys report the exact date of birth of the child and data on a variable $Demp$ that takes the value 1 if the mother was employed at the time of child birth, 0 otherwise. They also report the weight of each child which will be used to calculate the health indicator of interest: $\frac{Weight}{Age}$ where Age is measured in weeks. Let us denote this health indicator by Y_{it}. The survey also contains information on household demographics and socio-economic status of the child's household (denote the vector of these variables by Z_{it}). Suppose we estimate the following model by OLS.

$$Y_{it} = \beta_0 + \beta_1 t + \epsilon_{it} \tag{2}$$

(a) Suppose all women are employed at the time of child birth.

i. What is the interpretation of β_1? **(1)**

ii. Suppose (only for this question) we know that β_1 is consistently estimated by OLS. Is there a reason to add Z_{it} to the model? **(3)**

iii. It is well known that survey based date of birth data in India is an unreliable measure. Sometimes it is incorrect by even a few months. However this error is likely to be random. What is implication of this on the properties of your OLS estimator of β_1? **(3)**

iv. In the regression above suppose you add Age of the child as a regressor. Suppose that the problems with date of birth still apply (like the previous question). What is implication of this on the properties of the OLS estimator of β_1? **(3)**

v. Now assume that the date of birth has no error but we DO NOT know (unlike in the first question), that β_1 is consistently estimated by OLS. What can be the potential problems in interpreting the estimated coefficient as the effect of the maternity act of child health? How can inclusion of Z_{it} address some part of the problem? Are there any concerns even after inclusion of Z_{it}? **(3)**

(b) Suppose now that your dataset is slightly different. Not all women were employed at the time of child birth. You estimate the model with t and Z_{it} as regressors on children of all mothers, irrespective of whether they were employed or not. The health ministry official tells you that you are doing something wrong: the policy was meant for employed women. So you must run all your regressions only on children whose mother was employed at the time of child birth. You are not allowed to change your regression model but only allowed to run it on either your sample or the sample suggested by the ministry official. What will you do and why? **(5)**

(c) Suppose now that you can write a regression model that incorporates the whole sample but tries to address the concern of the ministry official. What specification will you estimate? **(3)**

(d) Will the OLS estimation of the specification suggested by you in the previous question lead to consistent estimation of parameters? Why or Why not? **(4)**