1. Suppose that $\{x_n\}_{n\geq 1}$ is a sequence of real numbers such that the inequality $|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|$ holds for all $n \geq 2$. Prove that $\{x_n\}_{n\geq 1}$ is convergent. [10]

2. Let $\mathbb{Z}$ be the set of all integers. Denote $\gcd(p, q)$ as the greatest common divisior of $p, q \in \mathbb{Z}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{p}{q^2}, & \text{if } x = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z},\ q \neq 0 \text{ and gcd(p,q)=1;} \\ 0, & \text{otherwise.} \end{cases}$$

   Show that f is differentiable at $x = 0$. [10]

3. Show that the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5$ has a negative root. [10]

4. Are the vectors $l_1 = (1, 1, 0, -1), l_2 = (5, 5, 0, 0), l_3 = (0, 0, -1, -1)$ linearly independent? [10]

5. Let the random variable $X \sim$ Binomial (n, p). Show that Variance of X is maximum when $p = \frac{1}{2}$. [10]

6. Let X, Y, Z denote 3 jointly distributed random variables with joint density function

$$f(x, y, z) = \begin{cases} K(x^2 + yz), & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

   Find the value of the constant K. Determine the marginal distribution of Y. [6+4]

7. Let x_1 and x_2 be the two real roots of the quadratic equation $ax^2 + ax + c = 0$ where $a \neq 0$ and $c \neq 0$. Moreover, one root is the negative of the reciprocal of the other root, i.e., $x_1 + \frac{1}{x_2} = 0$. Find the values of x_1 and x_2. [10]

8. (a) A code has 4 digits in a specific order. The digits are between $0-9$ (including both 0 and 9). How many different codes are possible if each digit may only be used once and we cannot have a code such that 0 is the first digit?

   (b) In how many different ways can the word 'political' be rearranged such that the vowels figuring in that word (o, i, i, a) only occur in even places?

   (c) In how many different ways can the word 'policy' be rearranged such that the vowels figuring in that word (o, i) only occur in even places?

   [2+4+4=10]

9. Consider a consumer in a two-commodity economy whose utility function is given by $U(x_1, x_2) = A-(x_1)^2-(x_2)^2+2a_1x_1+2a_2x_2-(a_1)^2-(a_2)^2$ with $A > 0$, $a_1 > 0$, $a_2 > 0$, $x_1 \geq 0$ and $x_2 \geq 0$. Let $p_1 > 0$ be the price of commodity 1, $p_2 > 0$ be the price of commodity 2 and M be the money income of the consumer.

   (a) If $M > p_1a_1 + p_2a_2$, then what can you say about the optimal value of the Lagragian multiplier associated with the budget constraint $M \geq p_1x_1+p_2x_2$ when the consumer maximizes his utility subject to the budget constraint? Justify your answer.

   (b) If $M < p_1a_1 + p_2a_2$, then how does your result change?

   [5+5=10]

10. If a function $f : [0, \infty) \to [0, \infty)$ is concave and $f(0) \geq 0$, then show that $f(x + y) \leq f(x) + f(y)$ for all $x, y \in [0, \infty)$. [10]

There are four sections in this test: A, B, C and D. Answer 4 questions. At least one question has to be answered from each of the sections A, B and C. The fourth question can be answered from any section.

Group A

1. Suppose a producer uses capital and labour as inputs. Let capital be denoted by k, and labour by l. Suppose the production function is $y(k,l)$. Let w be the wage rate of labour and r be the rental rate of capital.

   (a) Provide an example of a production function y which is continuous everywhere in k and l, yet not differentiable everywhere in k and l, such that capital and labor are substitutes in production. Write the function in algebraic form.
   (b) For your example, draw the isoquant map using representative isoquants. Also represent the isoquant(s) algebraically.
   (c) For your example, how does the optimal capital choice depend on w and r?

   [5+10+10=25]

2. Consider an n-firm industry of a homogeneous product; $n \geq 2$. Firms play Cournot. Unit cost of production by each firm is $m > 0$. The market price for the product is $P = \max\{0, A - X\}$, where X is industry output and A is the demand shift parameter; $A > m$. Now consider the merger of a subset of firms. Assume that a merger of k firms will be profitable if the merged firm's profit is at least as large as the sum of profits of k independent firms in an n-firm oligopoly industry; $1 < k \leq n$. Answer the following.

[***Note**: if k firms merge in an n-firm Cournot oligopoly, the merged firm's profit is: $\pi(k,n) = \frac{(A-m)^2}{(n-k+2)^2}$. Don't derive payoffs separately.]*

   (a) Examine whether a merger of two firms (i.e., $k = 2$) in an n (≥ 2) firm industry is always profitable.
   (b) Derive the condition, in general, for a profitable merger of k firms; $2 \leq k \leq n$.
   (c) If there are exactly 5 firms in the industry (i.e., $n = 5$), how many firms, at the minimum, should merge to make it a profitable merger? Give the intuition behind your result.

   [6+10+9=25]

3. (a) An indivisible object is sold to 3 buyers. If any buyer i gets $q_i \in \{0,1\}$ quantity of the object and makes a payment of p_i, her payoff is $(q_i v_i - p_i)$, where v_i is the value of the object for buyer i. Suppose $v_1 > v_2 > v_3$. Payment p_i can be positive, negative or zero (some buyers may be paid or compensated).

Consider the following normal form game. The seller asks each buyer to place a bid. If (b_1, b_2, b_3) are the bids of the buyers then the highest bidder wins. Ties are broken in favour of higher indexed buyer: for instance, if $b_1 \geq b_i$ for all $i \neq 1$, then buyer 1 wins the object. If bidder i wins, she pays $\max_{j \neq i} b_j$. Out of this payment, to every buyer k $\in$ {1, 2, 3}, the seller returns (i.e., k is compensated)

$\frac{1}{3} \min_{j \neq k} b_j$;

if k is the highest bidder or the second highest bidder. If k is the lowest bidder, then she is compensated

$\frac{1}{3} \max_{j \neq k} b_j$.

Here the highest, the second highest, and the lowest bidders are determined by using the same tie-breaking rule as mentioned before - that is, ties are broken in favour of the higher indexed bidder. For instance, if $b_1 = b_2 = b_3$, then 1 is the highest bidder, 2 is the second highest bidder, and 3 is the lowest bidder; if $b_2 = b_3 > b_1$, then 2 is the highest bidder, 3 is the second-highest bidder, and 1 is the lowest bidder.

Show that bidding her own value is a weakly dominant strategy for each buyer.

(b) Suppose (σ_1^*, σ_2^*) is a Nash equilibrium mixed-strategy profile in a two-person normal form game. Either prove (with appropriate arguments) or provide a counterexample to the following statement: if a pure strategy s_1 is played with positive probability in σ_1^*, then s_1 is not strictly dominated.

[15=10=25]

Group B

1. Consider an agent who lives for two periods. She earns wage income only in the first period of her life by supplying her labour in inelastic manner. The timing of the events is as follows – the agent first receives the wage income W and then allocates her portfolio before any first period consumption. The agent has the following utility function at the beginning of her life:

$$U = x\ln(c_1) + \beta(1-x)\ln(c_2);$$

where β is a parameter with $0 < \beta < 1$, and x is a random variable which can take two values: 1 with probability p and 0 with probability $(1-p)$. The realization of x occurs in the first period after she allocates her resources but before making any consumption in the first period. The agent has two technologies to invest in: storage technology and investment technology. Under storage technology, one unit of investment of the consumption good at the beginning of the first period secures $R_s > 0$ units of gross return either later in the first period or in the second period. Investment technology on the other hand guarantees $R_i > 0$ gross return in second period but no return if liquidated early. So, precisely, storage technology is for liquid assets and investment technology is for illiquid assets. Let θ represent the fraction of savings put in the storage technology during allocation at the beginning of the first period.

   (a) Write down the expected utility maximization problem of the agent.

   (b) Derive the optimum level of θ, that is, the value of θ that maximizes expected utility. Under what condition on p is the value of θ equal to 1?

   (c) Does optimal utility increase with $R_s > 0$? In either case show your result.

[9+(6+4)+6=25]

2. Consider an economy producing a single consumption good, Y. The aggregate profit-maximizing firm produces this good by the single factor of production, labour, L, according to the production function $Y = L^{\theta}$, where $0 < \theta < 1$. The representative household has labour income, wL (where L is the labour supplied by the household, and w is the wage rate), and an initial money holding M'. Its preferences are given by:

$$U = C^{\alpha}\left(\frac{M}{P}\right)^{1-\alpha} - L^{\gamma};$$

where $\gamma > 1$, $0 < \alpha < 1$; C is aggregate consumption, M is the money holding the household ends up with, P is the price of good Y and the last term, $-L^{\gamma}$, denotes the disutility from labour supply. The budget constraint of the household is given by:

$$PC + M = wL + M' + \pi;$$

where π is the household's profit income. In equilibrium, $Y = C$ and $M = M'$. Assume price and wage are both flexible.

(a) Solve for the competitive equilibrium output and employment.

(b) Find the effect on output of a monetary expansion (i.e., find the sign of $\frac{dY}{dM'}$).

[15+10=25]

3. Consider the Solow-Swan growth model. Assume that the production function is of the form

$$Y_t = F(K, AL);$$

where A is the level of technological progress and grows at the rate $g > 0$, L is population which grows at the rate $n > 0$, K is the capital stock and Y is GDP. Assume that

$$\dot{K} = sY - \delta K;$$

where the parameter $s \in [0,1]$ denotes the savings rate and the parameter

$\delta \in [0,1]$ denotes the depreciation on capital. Define $Z = \frac{K}{AL}$ as the capital labour ratio in efficiency units.

(a) Derive an expression for $\frac{\dot{Z}}{Z}$.

(b) Recall that $F(.,.)$ is linearly homogeneous in both its arguments. Noting that $f(Z) = F(Z,1)$ where $f(.)$ is the intensive form production function, show that for any $\lambda > 1$,

$$\frac{f(\lambda Z)}{\lambda Z} < \frac{f(Z)}{Z}.$$

Interpret this inequality in words.

(c) Use bullets (a) and (b) to draw a diagram describing the dynamics of growth in the Solow-Swan model. Plot Z on the x-axis and the appropriate functions on the y-axis, to show the dynamics of growth, $\frac{\dot{Z}}{Z}$, relative to the steady state, $\bar{Z}$

(d) Let $Q_t = \frac{Y_t}{L_t}$ denote output per worker. Show that on the balanced growth path,

$$\ln(\bar{Q}_t) = \ln(A_t f(\bar{Z})) + gt,$$

where $\bar{Z}$ is the steady state value of Z and $\bar{Q}$ is the balanced growth path value of Q. Draw a diagram plotting $\ln(Q_t)$ on the y-axis and time (t) on the x-axis with both (i) $g > 0$ and (ii) $g = 0$. Describe the growth implications across these two cases in a couple of sentences. India's savings rate has fallen during COVID-19. In a separate diagram, show the effects of a fall in the savings rate on India's growth in the transition and the long run using this diagram (under the assumption that $g > 0$).

[4+5+8+8=25]

Group C

1. An economist is asked to evaluate whether having science, technology, engineering and math institutions (STEM) in a district of India fosters demand for science education among high school children living in that district. She has access to two data sets for the year 2011: (1) a data set that lists the STEM institutes in each district of India, and (2) a data set that is based on a simple random sample of children of ages 17 to 18 years in 2011; the data set also contains information on the high school stream choice (Science, Commerce and Arts) of all the surveyed children who are enrolled in high school. The data set also provides indicators of socio-economic characteristics of the households the children belong to. Suppose that all the children in the surveyed age group are enrolled in high school. Consider the model

$$Y_{id} = \beta_0 + \beta_1 STEM_d + \varepsilon_{id}; \quad (1)$$

where i denotes a child and d denotes a district. Y is a binary variable that takes the value 1 if the child has chosen science stream and 0 otherwise; $STEM$ denotes the number of $STEM$ institutions in a district.

   (a) If the economist estimates β_1 using Ordinary Least Squares (OLS), when will it be consistent? Explain intuitively why it is likely to be inconsistent. (Do not restate the technical condition in words.)

   (b) In light of the inconsistency, the economist decides to add some control variables. One set of candidate variables for inclusion are socio-economic characteristics (denote by Z), which will typically have an important role to play in high school stream choice. These variables may have some correlation with the number of STEM institutes in a district, but this correlation is likely to be low. Moreover, the economist also collects data on some district level variables like Urbanization rate, Population, Literacy (denoted by *DIST*). These variables will have some small direct effect on the stream choice of students but are likely to be highly correlated to the number of STEM institutes in a district. Explain why the inclusion of *DIST* is more important for the consistency of the estimated β_1 than the inclusion of *Z*. Why is it important to still include *Z*? (NOTE: we are NOT interested in predicting *Y*.)

   (c) Someone states that since there are differences in state policies regarding education, one should account for them by including state level dummy variables. Rewrite the empirical model to account for such differences across the 28 states of India. Explain how the inclusion of these dummy variables accounts for the differences across states.

(d) Suppose the economist estimates any of the above specifications. She feels that she can provide *proof* that her estimator is consistent by the following procedure: after estimating the model by OLS, she regresses the residual of the regression on *STEM*. She contends that if the R^2 of this regression is 0 then the error term does not co-vary with STEM. Prove that this is an incorrect argument.

(e) Someone points out that it is getting harder nowadays to classify high school stream choice into the stated three categories because students take up many combinations that cut across these traditional streams. But the survey questionnaires have not been updated to reflect this. So the surveyed students will be required to choose one of the three options: hence there will be some mismeasurement of stream choice. What will be the impact of this on properties of the estimated β_1?

[5+10+3+5+2=25]

2. Consider the following time series:

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1},$$

where ε_t is a white noise process with mean 0, variance σ^2; and all the autocorrelations of ε_t are zeros.

(a) Examine if the time series Y_t is stationary or not.

(b) Derive the condition such that Y_t is invertible.

(c) Define Z_t=$Y_t + X$, where X and Y_t are independent for all time points – past, present and future; $X \sim N(0, 10)$ and Y_t is as defined above. Find the autocorrelation function of Z_t. Examine whether Z_t is stationary.

[6+6+13=25]

3. A fictional island country named NOWHERELAND shut itself from the world before a single Covid infection was detected. In January of 2021, it procured enough doses of a single dose vaccine to vaccinate half its population. Thereafter no more vaccination was done but the economy was opened - predictably many cases of Covid-19 came up. In May 2021, the health ministry conducted a survey of all individuals in the country. The survey collected information on vaccination status, demographic variables and an indicator of whether the person worked or not (a binary variable). The ministry wants to estimate the benefits of vaccination: in particular, it wants to contend that the vaccine had positive effects on an individual's labour supply. You are the analyst.

   (a) Suppose vaccination centres gave vaccines to people, who came voluntarily, on a first come first served basis - till they exhausted the vaccines they had (PROTOCOL 1).

      (i) Write down an empirical model, to estimate by Ordinary Least Squares, the impact of getting a vaccine on an individual's labour supply. Under what condition(s) will the estimated parameter be the true impact of vaccination?

      (ii) Suppose you discover that the ministry conducted a similar survey in November 2020 (for obvious reasons the question on vaccine status was not asked). You therefore have two observations on each person. Will this help you to come up with a better specification to estimate the impact of vaccination? If yes, write down the empirical model and explain why. If not, with the help of the same empirical model, explain why not.

   (b) Suppose vaccination centres now followed a different protocol (PROTOCOL 2). They allowed vaccines only to people 45 years old and above. The modality was the same as above – first come first served - and getting a vaccine shot was voluntary. Can you use this age cut-off to suggest a method to estimate the impact of the vaccine on labour supply? You have access to both the surveys mentioned above.

   (c) Now suppose that NOWHERELAND had followed a completely different protocol. It had listed down all the households and used a random number generator to draw up the list of the individuals (half the population) that should be vaccinated (PROTOCOL 3). The individuals eligible for the vaccine (in the list) could however choose not to take it. Using the same surveys how would you estimate the impact of vaccination on labour supply? Write down the empirical model and describe how estimation under this protocol may lead to better estimates than in the previous two protocols.

[(5+8)+6+6=25]

Group D

1. Carbon dioxide emissions vary across countries and across individuals within countries.

   (a) Which countries are the three largest emitters of CO2 per year?

   (b) What are the main determinants of annual CO2 emissions of countries? What about emissions per capita? Explain the reasons why the factors you have listed explain cross-country variation in CO2 emissions per capita.

   (c) Which variable accounts the most for variation in CO2 emissions across individuals within countries?

   (d) What policies do you think governments across the world, and in India in particular, could adopt to reduce CO2 emissions, or at least reduce increases in CO2 emissions, cost-effectively?

   (e) How could these policies be designed or modified to be pro-poor while still being cost-effective?

   [3+8+1+8+5=25]

2. (a) What was the share of health expenditure in the GDP of India before the recent pandemic? What percentage of health spending was public (i.e., by government?)

   (b) How do these compare to the global averages and to other lower-middle income countries?

   (c) What are some of the reasons why the share of health expenditure in India compares to the global average in the way that it does?

   (d) Some high-income countries have almost complete public provision of health care – like the UK, some have mostly private health providers but nationalized health insurance, so that all health services are paid for, but not provided by, the government (as in Canada); and some have private health care and private health insurance but with regulations and financial support to ensure that everyone is covered (Germany and Switzerland). How can good quality and sufficient health care be provided to everyone in India? Explain how you think India should move towards this goal and how it can be financed. Provide reasons why you think this is the best path forward.

   [2+4+9+10=25]