Sample Exam for 2012

Booklet No. TEST CODE: REI

Forenoon

Questions: 9	**Time: 2 hours**

- On the answer booklet write your Name, Registration number, Test Code, Number of the booklet etc. in the appropriate places.
- The test has 9 questions. **Answer any five**. All questions carry equal (20) marks.

1. (a) Prove by induction or otherwise

$$\frac{\left(\frac{5}{4}\right)^n}{n!} < \frac{1}{2^{n-2}} \text{ for } n = 2, 3, \ldots \textbf{ (10 marks)}$$

(b) Use this to show that $\log_e 5 > \frac{5}{4}$. **(10 marks)**

2. (a) For any finite set X, let $|X|$ denote the number of elements in X. Suppose A, B, C are three finite sets with $|A| = 8$, $|B| = 10$, $|C| = 12$, $|A \cup B| = 14$, $|A \cup C| = 14$, $|B \cup C| = 17$, $|A \cup B \cup C| = 17$. Find $|A \cap B \cap C|$. **(12 marks)**

(b) Define for any sets X, Y, $X \setminus Y = \{x \in X : x \notin Y\}$ (i.e., $X \setminus Y$ consists of all elements of X which are not in Y) and $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. In 3(a), find $|A \triangle B|$. **(8 marks)**

3. Convex Sets

(a) Consider the following system of linear equations with three variables x_1, x_2, x_3.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned}$$

Here, a_{ij} with $i, j \in \{1, 2, 3\}$ and b_1, b_2, and b_3 are real numbers. Suppose $(1, 1, 0)$ and $(0, 0, 1)$ are solutions to this system of equations. Is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ a solution to this system of linear equations? Explain your answer. **(5 marks)**

(b) Let $S_1, S_2 \subseteq \mathbb{R}^n$ be two convex sets. For each of the following sets, either prove that it is convex or give a counterexample.

(i) $S_1 + S_2 = \{z \in \mathbb{R}^n : z = x + y, x \in S_1, y \in S_2\}$. **(5 marks)**

(ii) $S_1 \cup S_2$. **(5 marks)**

(iii) $S_1 \cap S_2$. **(5 marks)**

4. (a) There are three cards. The first is green on both sides, the second is red on both sides and the third is green on one side and red on the other. We choose a card at random and we see one side (also chosen at random). If the side we see is green, what is the probability that the other side is green? **(12 marks)**

   (b) Let x be a random variable with cumulative distribution function F. Suppose F is strictly increasing. Consider $y = F(x)$. For any $a \in [0, 1]$, find Probability$(y < a)$. Based on this, what distribution does y follow? **(8 marks)**

5. (a) Find the maximum of the function $f(x) = x^{\frac{1}{x}}$ for all $x > 0$. **(15 marks)**

   (b) Find the maxima and the minima, if any, of the function $f(x) = x^3 - 6x^2 + 24x$ for all real values of x. **(5 marks)**

6. (a) Prove that $e^x = x + 2$ has a solution. **(10 marks)**

   (b) Three cards, say I, II, III, are lying on a table (see Figure 1). A number or a letter is printed on each side of every card.

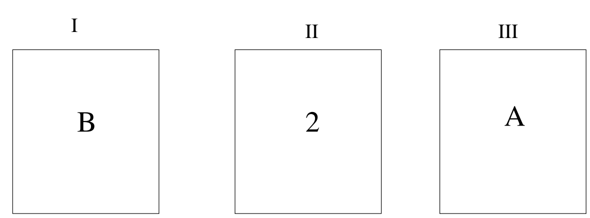

Figure 1: Three cards

Consider the statement

> If an even number is printed on one side of a card, then the letter "A" is printed on the other side.

Which cards would one have to turn over to check the truth of the statement? **(10 marks)**

7. (a) Consider the matrix given below.

$$\mathbf{A} = \begin{bmatrix} x & 0 & k \\ 1 & x & k-3 \\ 0 & 1 & 1 \end{bmatrix}$$

Suppose determinant of $\mathbf{A}$ is zero. What is the least positive integral value of k for which x can take real and more than one distinct values? **(10 marks)**

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave function. Let A be an $n \times m$ matrix, and let $b \in \mathbb{R}^n$. Consider the function $h : \mathbb{R}^m \to \mathbb{R}$ defined by

$$h\,(x) = f\,[Ax + b]\,,\ x \in \mathbb{R}^m.$$

Is h concave? Why or why not? Explain clearly. **(10 marks)**

8. Consider the following joint distribution function of random variables x and y

$$F_{xy} = P(x \leq t, y \leq s) = \begin{cases} 0 & t < 0 \text{ or } s < 0, \\ \frac{ts}{2} + \frac{\alpha t(1-t)s(2-s)}{4} & 0 \leq t \leq 1, 0 \leq s \leq 2, \\ t & 0 \leq t \leq 1, s \geq 2, \\ \frac{s}{2} & t \geq 1, 0 \leq s \leq 2, \\ 1 & t \geq 1 \text{ and } s \geq 2 \end{cases}$$

where α is some real number between -1 and 1.

Determine the marginal distribution functions F_x and F_y. For what value(s) of α are the random variables x and y independent. **(6+6+8 marks)**

9. (a) Consider the following function

$$f(x) = \begin{cases} -ae^x & x < 0 \\ a^2 - 1 & x \geq 0. \end{cases}$$

where a is a constant. Find the real values of a for which f is continuous at $x = 0$. Is f differentiable at $x = 0$ for these values of a? Explain your answer. **(8+4 marks)**

(b) Find the solutions to the following optimization problem. **(8 marks)**

$$\max_{x_1, x_2 \in \mathbb{R}} x_1 x_2$$

$$\text{s.t.}$$

$$x_1^2 + x_2^2 \leq 16.$$

Sample Exam for RE II, 2012

The test has 10 questions. **Answer any four**.

All questions carry equal marks.

1. You want to study the effect of education on income for which you estimate a relationship between years of education and the log of income as shown in Figures 1 and 2. The dots are the data points for each individual in your sample and the line is the estimated relationship between years of education and log(income). Figure 1 shows the relationship for the full sample while in Figure 2 [log(income), education] is observed iff $y < y_p$ where y is log(income), and y_p is the poverty line.

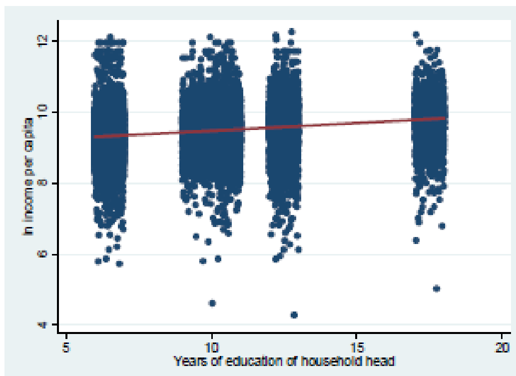

Figure 1: Full Sample

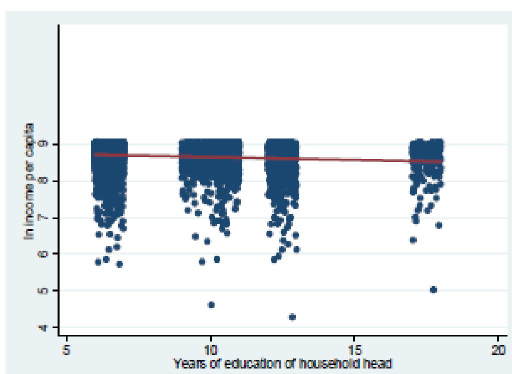

Figure 2: Below Poverty Line Sample

(a) Interpret the effect of education on income as estimated in 1 and 2. [50 words]

(b) Does one figure give you a more reliable estimate of the relationship between eduction and income than the other? Justify your answer. [50 words]

(c) Mathematically derive the relationship between years of education (x) and the log of income (y) in the sample shown in Figure 2, that is, express $E[y|x, y < y_p]$ given that you want to estimate the relationship $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$. Explain how this relates to your answer in part (b) above. [50 words]

2. The government has instituted an employment guarantee scheme (EGS) in rural areas, beginning in 2010, which aims to alleviate poverty. The program guarantees employment or commits to providing employment at a fixed wage-rate for 100 days in a year to any one member of a household which demands employment. Access to the program is universal, (not means-tested), one must work to get paid and the wage rate is set at a centrally determined minimum level.

   (a) Briefly identify the main challenge in evaluating the causal impact of this program on poverty alleviation at the household level if you have cross-sectional data on a random sample of households for 2011. [50 words]

   (b) Suppose you could have designed a pilot program of the EGS in 2009 before the government implemented the full program. Can you suggest a program implementation strategy that you can adopt in this pilot to overcome the challenge you mentioned in part (a)? [200 words]

3. Table 1 shows the regression output of education expenditure on children going to school on the following list of X variables (the unit of observation is a child). **Note**: The average age of children in your sample is 10.

   Do not calculate any numbers. Just write out the expressions till the stage before you need to calculate.

   (i) What variables are significant at 0.05 level?

   (ii) Interpret the coefficient of the constant.

Education Expenditure	Coeff.	Std. Err.	$\frac{\text{Coeff}}{Std.Err.}$	p value
$D_{\text{male}} = 1$ if male child, zero otherwise	37.18	5.96	6.23	0.00
Age of Child	84.51	12.62	6.70	0.00
Square of Age of a child	7.50	0.70	10.69	0.00
$D_{\text{headmale}} = 1$ if Household head is a Male and zero otherwise	-133.99	95	-1.41	0.12
$D_{\text{SC}} = 1$ if Child is from Schedule Caste household and zero otherwise	-55.37	7.79	-7.11	0.00
Constant	358.39	55.99	6.40	0.00

Table 1: Regression output of Education Expenditure on school children - Last column shows p value for the null Hypothesis that coefficient =0

(iii) What is the marginal effect of age of the child on education expenditure. If a policy maker asked you for one number to capture this marginal effect, what would be your answer?

(iv) Does it matter that a child is from a schedule caste household? If yes, quantify the effect?

(v) The marginal effect in part (iv) is independent of age of the child. What term would you have to add to the specification to allow for dependence of (iv) on age of the child.

(vi) If the government was thinking of universal coverage through full education subsidy to lower economic levels and wanted to use your regression results, does the fact that that you have excluded children not in school matter (since for them education expenditure is zero)? Explain why or why not.

4. (a) A researcher estimates the following model based on 100 observations, for stock market returns, both thinks that there may be an econometric problem with it.

$$\underset{(0.436)}{y_t = 0.638} \quad + \quad \underset{(0.291)}{0.402x_{1t}} \quad + \quad \underset{(0.763)}{0.891x_{2t}}, \qquad R^2 = 0.97$$

where figures in parentheses are the standard errors of the corresponding coefficient estimates.

Suggest, along with justifications, what the econometric problem might be. Also indicate how you might go about solving the perceived problem. (The critical values of the standard normal distribution at 5 percent and 1 percent levels of significance are 1.96 and 2.57 respectively.)

(b)

(i) Consider the following regression model

$y_i = \alpha + \beta x_i + \epsilon_i.$

Suppose that the regressor is correlated with the error term. Would ordinary least squares (OLS) be a valid method of estimation in this case? Explain.

(ii) From a sample of 100 observations, the following data are obtained. Calculate the instrumental variable estimates of α and β where z is the instrumental variable.

$\sum y_i^2 = 350$, $\sum x_i y_i = 150$, $\sum z_i y_i = 100$, $\sum y_i = 100$, $\sum x_i^2 = 400$, $\sum z x_i = 200$, $\sum x_i = 100$, $\sum z_i^2 = 400$, and $\sum z_i = 50$.

5. Consider the normal form game played by two players as shown in Table 2. Player 's strategies are rows (U, C, D) and Player 2's strategies are columns (L, M, R). In each cell of Table 2, the first number denotes the payoff of Player 1 and the second number denotes the payoff of Player 2.

	L	M	R
U	$5, 5$	$0, 0$	$0, 6$
C	$0, 0$	$4, 4$	$0, 1$
D	$6, 0$	$1, 0$	$1, 1$

Table 2: A Normal Form Game

(i) Which pure strategy of Players 1 and 2 are strictly dominated? **(2 marks)**

(ii) Find the set of strategies that remain after iterative elimination of strictly dominated strategies. **(5 marks)**

(iii) Find the set of Nash equilibria (both pure and mixed) of this game. **(10 marks)**

(iv) Suppose the game is played *sequentially* with Player 1 moving first. Player 2 observes the move of Player 1 and moves next. Find the set of subgame perfect Nash equilibria of this extensive form game. **(8 marks)**

6. (a) A seller is selling a single indivisible good to a set of three buyers, denoted by $B = \{1, 2, 3\}$. The valuation of buyers for the good are given by $v_1 = 6, v_2 = 4, v_3 = 2$, where v_i denotes the valuation of buyer $i \in B$. Suppose the seller announces a price p. If buyer i buys the good at price p his utility is $v_i - p$. Not buying the good gives zero utility to every buyer. The demand set of every buyer $i \in B$ at a price p is defined as follows:

$$D_i(p) = \begin{cases} \{0\} & \text{if } v_i - p < 0 \\ \{1\} & \text{if } v_i - p > 0 \\ \{0, 1\} & \text{if } v_i - p = 0 \end{cases}$$

A buyer $i \in B$ is a **serious demander** at price p if $D_i(p) = \{1\}$. We say there is **overdemand** at price p if the number of serious demanders at p is more than one. We say there is **underdemand** at price p if for every buyer $i \in B$ we have $D_i(p) = \{0\}$.

(i) For prices $0, 1, 2, \ldots, 6, 7$ write down the demand set of every buyer, the set of serious demanders, whether there is overdemand or not, and whether there is underdemand or not. **(8 marks)**

(ii) Suppose there are n ($n \geq 2$) buyers with values $v_1, v_2, \ldots, v_n$ and $v_1 \geq v_2 \geq \ldots \geq v_n$. What is the maximum price below which there is always overdemand. What is the minimum price above which there is always underdemand. What is the range of price where there is no overdemand and no underdemand. Explain your answers. **(3+3+3 marks)**

(b) A TV show holds the following contest. Every viewer is asked to send in an integer in $[0, 100]$ to the show. Based on the integers received, the show defines a **pivotal integer** x as

$$x = \lfloor \frac{\max_i y_i}{2} \rfloor,$$

where $y_1, y_2, \ldots$ denotes the entries in the show and for every real number z, $\lfloor z \rfloor$ gives the largest integer $\bar{z}$ such that $z \geq \bar{z}$.

The viewers who send the pivotal integer as their entry share the prize money equally amongst themselves. Suppose the utility of a viewer is the prize money she wins. Which integer do you think a viewer must send to maximize her utility? Find a Nash equilibrium of this game. **(4+4 marks)**

7. With capital k fixed, the short-run variable cost of producing output level y for a firm is:

$$wy\left(\frac{a}{k} + m\right)$$

where w is the price of the variable input (labor), and a and m are are constants. The interest cost of capital per period is r per unit of capital. Thus, short-run total costs are:

$$c(w, r, y, k) = wy\left(\frac{a}{k} + m\right) + rk$$

(i) In the long run k is not fixed, and the firm can choose it optimally. Derive the long-run total cost function $c(w, r, y)$ for this firm, and show that it has declining average and marginal costs.

(ii) Find this firm's production function $f(k, l)$, by using the fact that the short-run variable cost is, by definition, equal to

$$\min_l wl \text{ subject to } f(k, l) \geq y$$

8. Consider a closed economy in which a household's (L^s) labor supply to firms is determined by the amount which maximizes utility, U, where

$$U = C^\alpha (1 - L^s)^\beta,$$

and C is the household (real) consumption expenditure, which is taken to be equal to its wage income (i.e., real wage multiplied by labor). Assume that $\alpha > 0, \beta > 0$, and the total amount of labor is normalized to unity.

(a) Find the first order condition for utility maximization and derive the household's labor supply (L^s) for a given real wage, w. Does L^s depend on w? Explain your answer.

(b) Assume this economy is a Keynes-style economy in which real investment expenditure, I, is autonomous, and output, Y,is determined by aggregate demand, i..e, $Y = C + I$. The aggregate production function is given by

$$Y = AL^{\theta}$$

where $A > 0$ and $0 < \theta < 1$. Find the equilibrium value of Y for a given value of I. What sort of role does θ play here. Give some intuitive explanation for your answer.

9. (a) Briefly explain the meaning of the term 'crowding out' of a fiscal stimulus in macroeconomics. Show your answer diagramatically.

   (b) Briefly explain why and when, under rational expectations, anticipated policy changes may affect real GDP.

10. Let the production function be

$$Y = K^{\alpha}(AL)^{\beta}R^{1-\alpha-\beta} \tag{1}$$

where Y denotes aggregate output, K denotes the aggregate stock of physical capital, L denotes labor, A is an index of technological progress, and R is the amount of land (which is available in finite supply). Assume, $\alpha > 0, \beta > 0$, and $\alpha + \beta < 1$. The factors of production evolve according to $\dot{K} = sY - \delta K$, $\dot{A} = Ag$, $\dot{L} = Ln$, and $\dot{R} = 0$, where $0 < \delta < 1$ is the prediction rate on physical capital, $s \in [0, 1]$ is the exogenous savings rate of the economy, $g > 0$ is the exogenous growth rate of technological progress, and $n > 0$ is the population growth rate. For some variable x, denote $\frac{dx}{dt} = \dot{x}$ and $\frac{\dot{x}}{x} = g_x$.

(i) Show that there are two possible values (one positive and one negative) for g_K. **(10 marks)**

(ii) Does this economy have a unique and stable balanced growth path? If so, what are those growth rates? If not, why not. **(5 marks)**

(iii) Does the stock of land being constant imply that permanent growth is not possible? Answer this using the positive value for g_K by deriving an expression for $\frac{\dot{Z}}{Z}$, where $Z = \frac{Y}{L}$, in terms of some of the parameters in the model. **(10 marks)**