Microeconomic Theory Solved Problems[1]
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Solved problems

1. Derive the Walrasian demands when the utility function is

   (a) $u(x_1, x_2) = x_1^\alpha x_2^\beta$
   Clearly, if either x_1 or x_2 is 0, utility is 0. This cannot be optimal since positive utility is possible. Hence we know that both must be strictly positive at the optimum so we can use a Lagrangian. The Lagrangian is
   $$\mathcal{L} = x_1^\alpha x_2^\beta + \lambda[w - p_1x_1 - p_2x_2]$$
   The first order conditions, then, are
   $$\alpha x_1^{\alpha-1} x_2^\beta - \lambda p_1 = 0$$
   and
   $$\beta x_1^\alpha x_2^{\beta-1} - \lambda p_2 = 0$$
   and, of course, the budget constraint.
   $$p_1x_1 + p_2x_2 = w$$
   We can solve the first equation for λ and substitute into the second to get
   $$\frac{\alpha x_1^{\alpha-1} x_2^\beta}{p_1} = \frac{\beta x_1^\alpha x_2^{\beta-1}}{p_2}$$
   or
   $$p_1x_1 = \frac{\alpha}{\beta} p_2x_2$$
   Substituting this into the budget constraint yields
   $$p_2x_2[1 + \frac{\alpha}{\beta}] = w$$
   or
   $$x_2 = \frac{\beta w}{p_2[\alpha + \beta]}$$
   and
   $$x_1 = \frac{\alpha w}{p_1[\alpha + \beta]}$$
   So,
   $$x(p, w) = (x_1, x_2)(p_1, p_2, w) = \left(\frac{\alpha w}{p_1[\alpha + \beta]}, \frac{\beta w}{p_2[\alpha + \beta]}\right)$$

   (b) $u(x_1, x_2) = x_1 + 2\sqrt{x_2}$
   Let's make good 1 the numeraire so that its price is 1 and let p_2 be the price of 2. The first order condition for an interior max is
   $$p_2 = \frac{1}{\sqrt{x_2}} \text{ i.e. } x_2 = \frac{1}{p_2^2} = p_2^{-2}$$
   What about corner solutions? There will never be a corner solution where $x_2 = 0$, since the marginal utility of x_2 approaches infinity as x_2 approaches 0. But there will be a corner solution with $x_1 = 0$ if $p_2(p_2^{-2}) = p_2^{-1} > w$ or equivalently if $p_2 < 1/w$. Hence, the demand is given by
   $$x(p, w) = (x_1, x_2)(p, w) = \begin{cases} (0, \frac{w}{p_2}) & \text{if } w < \frac{1}{p_2} \\ (w - \frac{1}{p_2}, \frac{1}{p_2^2}) & \text{if } w \geq \frac{1}{p_2} \end{cases}$$

[1]Please refer a good textbook in Microeconomics for the detailed treatment of the concepts. Few of those are listed here: Intermediate Microeconomics by Varian, Microeconomic Analysis by Varian, A Course in Microeconomic Theory by Kreps, Advanced Microeconomic Theory by Jehle and Reny, Microeconomic Theory by Mas-collel, Whinston and Green.
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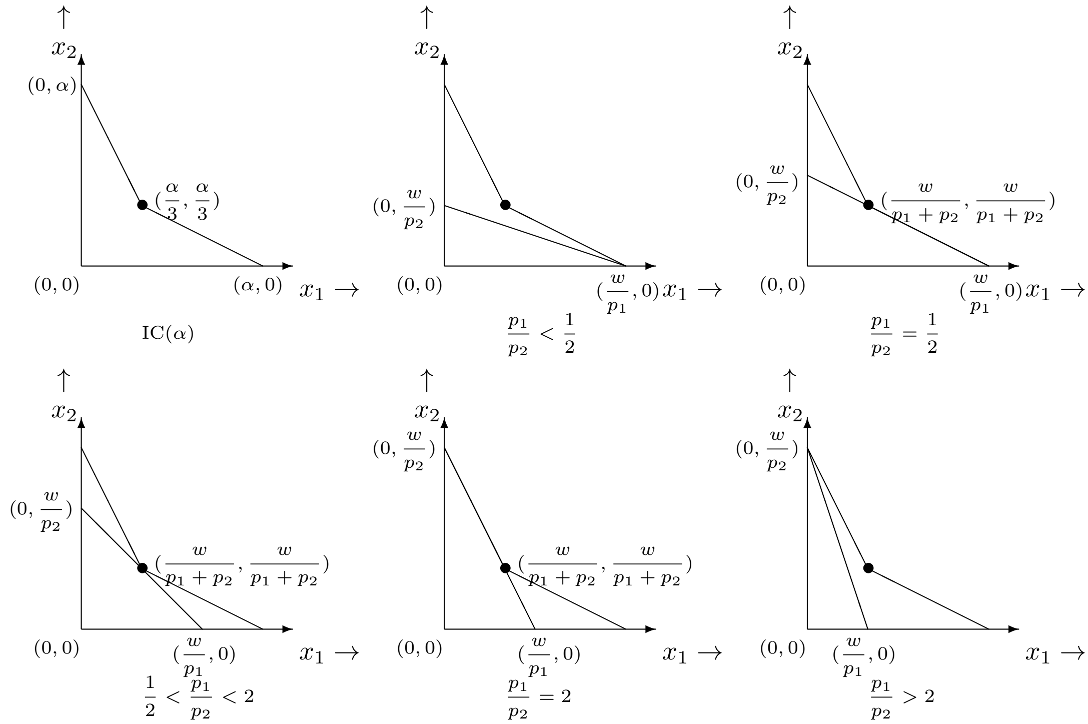
(c) $u(x_1, x_2) = \min\{x_1 + 2x_2, 2x_1 + x_2\}$

Lets plot an indifference curve for say $u = \alpha > 0$ and examine all the equilibrium possibilities,

$$\text{IC}(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2_+ \,|\, \min\{x_1 + 2x_2, 2x_1 + x_2\} = \alpha\}$$



We can summarize the above information to get the demand in the following way

$$x(p, w) = (x_1, x_2)(p, w) \in \begin{cases} \{(\frac{w}{p_1}, 0)\} & \text{if } \frac{p_1}{p_2} < \frac{1}{2} \\ \{(y_1, y_2) \in \mathbb{R}^2_+ | p_1 y_1 + p_2 y_2 = w, \frac{w}{p_1 + p_2} \le y_1 \le \frac{w}{p_1}\} & \text{if } \frac{p_1}{p_2} = \frac{1}{2} \\ \{(\frac{w}{p_1 + p_2}, \frac{w}{p_1 + p_2})\} & \text{if } \frac{1}{2} < \frac{p_1}{p_2} < 2 \\ \{(y_1, y_2) \in \mathbb{R}^2_+ | p_1 y_1 + p_2 y_2 = w, 0 \le y_1 \le \frac{w}{p_1 + p_2}\} & \text{if } \frac{p_1}{p_2} = 2 \\ \{(0, \frac{w}{p_2})\} & \text{if } \frac{p_1}{p_2} > 2 \end{cases}$$

2. Laxmi is a poor agricultural worker. Her consumption basket comprises three commodities: rice and two vegetables - cabbage and potato. But there are occasionally very hard days when her income is so low that she can afford to buy only rice and no vegetables. However, there never arises a situation when she buys only vegetables and no rice. But when she can afford to buy vegetables, she buys only one vegetable, namely the one that has the lower price per kilogram on that day. Price of each vegetable fluctuates day to day while the price of rice is constant. Write down a suitable utility function that would represent Laxmi's preference pattern. Explain your answer.
Check that the following utility function represents the preferences of Laxmi:

$$u(x_c, x_p, x_r) = \max\{x_c, x_p\} + \sqrt{x_r}$$

3. A monopolist has contracted to sell as much of his output as he likes to the government at Rs. 100/- per unit. His sale to the government is positive. He also sells to private buyers at Rs. 150/- per unit. What is the price elasticity of demand for the monopolist's products in the private market?
Since monopolist's sale to the government is positive, his marginal revenue at the point of sale in the private market must be Rs. 100/-. Now price in the private market is Rs. 150/-. We can compute the price elasticity of demand in the following way:

$$\text{TR}(x) = p(x) \cdot x$$

Differentiating TR(x) w.r.t. x, we get

$$\begin{aligned} & \text{MR}(x) = p(x) + x\frac{dp(x)}{dx} \\ \Leftrightarrow\quad & \text{MR}(x) = p(x) + p(x)\frac{x}{p(x)}\frac{dp(x)}{dx} \\ \Leftrightarrow\quad & \text{MR}(x) = p(x)\left(1 + \frac{1}{\xi}\right) \end{aligned}$$

Now substituting $p(x) = 150$ and MR$(x) = 100$ in above we get elasticity, $\xi = -3$

4. (ME-2.3(2008)) Two firms 1 and 2 sell a single, homogeneous, infinitely divisible good in a market. Firm 1 has 40 units to sell and firm 2 has 80 units to sell. Neither firm can produce any more units. There is a demand curve: $p = a - q$, where q is the total amount placed by the firms in the market. So if q_i is the amount placed by firm $i \in \{1, 2\}$, $q = q_1 + q_2$ and p is the price that emerges. a is positive and a measure of market size. It is known that a is either 100 or 200. The value of a is observed by both firms. After they observe the value of a, each firm decides whether or not to destroy a part of its output. This decision is made simultaneously and independently by the firms. Each firm faces a constant per unit cost of destruction equal to 10. Whatever number of units is left over after destruction is sold by the firm in the market. Solve for Nash equilibrium for both values of a.
   Firm 1's objective:

$$\begin{array}{cc} \max\limits_{q_1} & (a - q_1 - q_2)q_1 - 10(40 - q_1) \\ \text{subject to} & 0 \leq q_1 \leq 40. \end{array}$$

   Differentiating the objective we get,
   $(a - 2q_1 - q_2) + 10$
   Given q_2, if $(a - 2q_1 - q_2) + 10 \geq 0$ at $q_1 = 40$, then the best response of firm 1 is 40.
   If $(a - 2q_1 - q_2) + 10 \leq 0$ at $q_1 = 0$ then the best response of firm 1 is 0.
   If $(a - 2q_1 - q_2) + 10 = 0$ at some $0 \leq q_1 \leq 40$ then the best response of firm 1 is $q_1 = (a + 10 - q_2)/2$. To summarize, best response correspondence of firm 1 is:

$$\begin{array}{rll} q_1(q_2) & = 40 & \text{if } (a - 70 \geq q_2) \\ & = 0 & \text{if } (a + 10 \leq q_2) \\ & = (a + 10 - q_2)/2 & \text{if } (a - 70 \leq q_2 \leq a + 10) \end{array}$$

   Firm 2's objective:

$$\begin{array}{cc} \max\limits_{q_2} & (a - q_1 - q_2)q_2 - 10(80 - q_2) \\ \text{subject to} & 0 \leq q_2 \leq 80. \end{array}$$

   Similar to above, best response correspondence of firm 2 is:

$$\begin{array}{rll} q_2(q_1) & = 80 & \text{if } (a - 150 \geq q_1) \\ & = 0 & \text{if } (a + 10 \leq q_1) \\ & = (a + 10 - q_1)/2 & \text{if } (a - 150 \leq q_1 \leq a + 10) \end{array}$$

   Put $a = 200$, the Nash equilibrium is: $(q_1, q_2) = (40, 80)$
   Put $a = 100$, the Nash equilibrium is: $(q_1, q_2) = (110/3, 110/3)$

5. (ME-2.3b(2007)) Consider an IS-LM model for a closed economy with government where investment (I) is a function of rate of interest (r) only. An increase in government expenditure is found to crowd out 50 units of private investment. The government wants to prevent this by a minimum change in the supply of real money balance. It is given that $dI/dr = -50$, dr/dy (slope of the LM curve) $= 1/250$, and slope of the IS curve, $dr/dy = -1/125$ and all relations are linear. Compute the change in y from the initial to the final equilibrium when all adjustments have been made.

$$\begin{aligned} I &= A - 50r \\ \frac{M}{P} &= ky - hr = k(y - 250r)\ldots\text{because } dr/dy = k/h = 1/250 \\ (1 - c)y &= A - 50r + G\ldots\text{Since } dr/dy = -(1 - c)/50 = -1/125\text{, this gives us } c = 0.6 \\ \text{Thus } y &= 2.5A - 125r + 2.5G \end{aligned}$$

Since an increase in government expenditure is found to crowd out 50 units of private investment. This implies r has gone up by 1 as a result of increase in G. Solving IS-LM for r, we get:

$$\Delta r = \frac{1}{150}\Delta G$$

Putting $\Delta r = 1$, we get $\Delta G = 150$. Solving IS-LM for y, we get:

$$\Delta y = \frac{5}{3}\Delta G = 250$$

The above solution is not adjusted for the money supply change to rectify the crowding out: So we need to increase the money supply such that rate of interest again fall back to its original value i.e. we need to do the following: we will move along the new IS curve in such a way that r falls by 1 from its new value. Given the slope of the IS curve

$$dr/dy = -1/125$$

we get,

$$\Delta y = 125$$

So the total change in income due to fiscal policy and corresponding monetary policy is 250+125 = 375.

1. (Ref: DSE 2010(21)) Let N be the number of observations that lie in [a,b] when a sample (x_1, x_2) is drawn. Expected value of N is

$$\begin{aligned}
E(N) &= 0 \cdot \Pr(x_1 \notin [a,b], x_2 \notin [a,b]) + 1 \cdot \Pr(x_1 \in [a,b], x_2 \notin [a,b]) + \\
&\quad 1 \cdot \Pr(x_1 \notin [a,b], x_2 \in [a,b]) + 2 \cdot \Pr(x_1 \in [a,b], x_2 \in [a,b]) \\
&= 0 + 1 \cdot \Pr(x_1 \in [a,b]) \cdot \Pr(x_2 \notin [a,b]) + 1 \cdot \Pr(x_1 \notin [a,b]) \cdot \Pr(x_2 \in [a,b]) + \\
&\quad 2 \cdot \Pr(x_1 \in [a,b]) \cdot \Pr(x_2 \in [a,b]) \ldots \text{since the draws are independent} \\
&= 1 \cdot \int_a^b f(x)dx \cdot \left(1 - \int_a^b f(x)dx\right) + 1 \cdot \left(1 - \int_a^b f(x)dx\right) \cdot \int_a^b f(x)dx + 2 \cdot \int_a^b f(x)dx \cdot \int_a^b f(x)dx \\
&= 2 \cdot \int_a^b f(x)dx \cdot \left(1 - \int_a^b f(x)dx\right) + 2 \cdot \int_a^b f(x)dx \cdot \int_a^b f(x)dx \\
&= 2 \cdot \int_a^b f(x)dx
\end{aligned}$$

2. (Ref: DSE 2010(22)) Lets first find the distribution of $Y = \max\{X_1, X_2, \ldots, X_n\}$. For $0 < y < 1$,

$$\begin{aligned}
\Pr(Y \le y) &= \Pr(\max\{X_1, X_2, \ldots, X_n\} \le y) \\
&= \Pr(X_1 \le y, X_2 \le y, \ldots, X_n \le y) \\
&= \Pr(X_1 \le y) \cdot \Pr(X_2 \le y) \cdots \Pr(X_n \le y) \ldots \text{by independence of } X_i\text{'s} \\
&= y^n
\end{aligned}$$

Probability density of Y, denoted by $f(\cdot)$, is the derivative of y^n with respect to y,

$$f(y) = n \cdot y^{n-1}$$

Expectation of Y is

$$\begin{aligned}
E(Y) &= \int_0^1 yf(y)dy \\
&= \int_0^1 y(n \cdot y^{n-1})dy \\
&= \int_0^1 ny^n dy \\
&= \left[\frac{n}{n+1} y^{n+1}\right]_0^1 \\
&= \frac{n}{n+1}
\end{aligned}$$

3. (Ref: DSE 2010(23)) Given that X takes values in $\{-1, 0, 1\}$ and $\Pr(X = -1) = \Pr(X = 0) = \Pr(X = 1) = 1/3$. Define $Y = X^2$. $\Pr(Y = 1) = \Pr(X = 1) + \Pr(X = -1) = 2/3$ and $\Pr(Y = 0) = \Pr(X = 0) = 1/3$. Clearly $E(X) = 0$ and $E(Y) = 2/3$. To determine the correlation between Y and X, we first compute the covariance between X and Y.

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = E(X^3) = (-1)^3(\frac{1}{3}) + (0)^3(\frac{1}{3}) + (1)^3(\frac{1}{3}) = 0$$

Since the covariance is 0, X and Y are uncorrelated. To check for independence consider the event $(X = 0, Y = 0)$.

$$\Pr(X = 0, Y = 0) = \Pr(X = 0) = \frac{1}{3} \neq \frac{1}{9} = \frac{1}{3} \cdot \frac{1}{3} = \Pr(X = 0)\Pr(Y = 0)$$

Hence X and Y are not independent.

4. (Ref: DSE 2010(26)) For $x \in [-1, 1]$,

$$\begin{aligned}
\Pr(X \le x) &= \Pr(\sin\theta \le x) \\
&= \Pr(\theta \le \sin^{-1} x) \ldots \text{because sine is an increasing function over } [-\pi/2, \pi/2] \\
&= \frac{1}{\pi}\left(\sin^{-1} x + \frac{\pi}{2}\right)
\end{aligned}$$

5. (Ref: DSE 2008(58)) It can be easily shown (graphically, or otherwise) that $x > 1 + \log x, \forall x \neq 1$. Thus, for $x = \log \pi$ we have

$$\begin{array}{rc} & \log \pi > 1 + \log \log \pi \\ \Leftrightarrow & e^{\log \pi} > e^{1+\log \log \pi} \\ \Leftrightarrow & \pi > e^1 e^{\log \log \pi} \\ \Leftrightarrow & \pi \log e > e \log \pi \\ \Leftrightarrow & \log e^{\pi} > \log \pi^e \\ \Leftrightarrow & e^{\pi} > \pi^e \end{array}$$

6. (Ref: DSE 2009(II(17))) Jai and Vijay are taking a exam in statistics. The exam has only three grades A, B and C. The probability that Jai gets a B is 0.3, the probability that Vijay gets a B is 0.4, the probability that neither gets an A, but at least one gets a B is 0.1. What is the probability that neither gets a C but at least one gets a B?
In what follows, I am going to write Jai's grade first, followed by Vijay's grade. For example, if I mention the sample point AB, then that would mean that Jai's grade is A and Vijay's grade is B.
Pr(Jai gets a B) = Pr(BA or BB or BC) = Pr(BA) + Pr(BB) + Pr(BC) = 0.3 —(1)
Pr(Vijay gets a B) = Pr(AB or BB or CB) = Pr(AB) + Pr(BB) + Pr(CB) = 0.4 —(2)
Pr(Neither gets an A, but at least one gets a B) = Pr(BC or BB or CB) = Pr(BC) + Pr(BB) + Pr(CB) = 0.1 —(3)
Pr(Neither gets an C, but at least one gets a B) = Pr(BA or BB or AB) = Pr(BA) + Pr(BB) + Pr(AB) = 0.6 (Add (1) and (2) and then subtract (3) from it)

7. (Ref: DSE 2010(II(21))) Let N denotes the number of observations in the sample that fall within the specified interval $[a, b]$. Since we are drawing a sample of size 2, we have:

$$N = \begin{cases} 0, & \text{with probability } (1 - \int_a^b f(x)dx)^2 \\ 1, & \text{with probability } 2(\int_a^b f(x)dx)(1 - \int_a^b f(x)dx) \\ 2, & \text{with probability } (\int_a^b f(x)dx)^2 \end{cases}$$

$$\begin{aligned} \mathbb{E}(N) &= (0)(1 - \int_a^b f(x)dx)^2 + (1)2(\int_a^b f(x)dx)(1 - \int_a^b f(x)dx) + (2)(\int_a^b f(x)dx)^2 \\ &= 2(\int_a^b f(x)dx) \end{aligned}$$

8. (ME-1.18(2010)) Let X be a Normally distributed random variable with mean 0 and variance 1. Let $\Phi(\cdot)$ be the cumulative distribution function of the variable X. Then the expectation of $\Phi(X)$ is
To find the Expectation of $\Phi(X)$, we will first recover its distribution: For $0 < a < 1$,

$$\begin{aligned} \Pr(\Phi(X) \leq a) &= \Pr(X \leq \Phi^{-1}(a)) \ldots \text{because } \Phi \text{ is strictly increasing} \\ &= \Phi(\Phi^{-1}(a)) \\ &= a \end{aligned}$$

Thus $\Phi(X)$ is distributed uniformly over $(0, 1)$. Hence the expectation of $\Phi(X)$ is 1/2.

9. (ME-1.6(2006)) Let $f(x)$ be a function of real variable and let Δf be the function $\Delta f(x) = f(x+1) - f(x)$. For $k > 1$, put $\Delta^k f = \Delta(\Delta^{k-1} f)$. Then $\Delta^k f(x)$ equals
For $k = 1$,
$\Delta^1 f(x) = \Delta f(x) = f(x+1) - f(x)$ (This will rule out options (a) and (d))
For $k = 2$,
$\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta f(x+1) - \Delta f(x) = f(x+2) - f(x+1) - f(x+1) + f(x) = f(x+2) - 2f(x+1) + f(x)$ (This will help us rule out option (b)).
So, the answer must be (c).

10. (ME-1.14(2006)) The number of positive pairs of integral values of (x, y) that solves $2xy - 4x^2 + 12x - 5y = 11$ is

$$\begin{aligned} 2xy - 4x^2 + 12x - 5y &= 11 \\ (2x - 5)y &= 4x^2 - 12x + 11 \\ y &= \frac{4x^2 - 12x + 11}{2x - 5} \\ y &= 2x - 1 + \frac{6}{2x - 5} \end{aligned}$$

Now 6 is divisible by $2x-5$ for the following values of x: $1, 2, 3, 4$. And y is positive when x is either 3 or 4. Hence the answer is 2.

11. Consider the function $f(x_1, x_2) = \int_0^{\sqrt{x_1^2+x_2^2}} e^{-(w^2/(x_1^2+x_2^2))} dw$ with the property that $f(0,0) = 0$. Show that the function is homogeneous of degree 1.

$$f(\lambda x_1, \lambda x_2) = \int_0^{\sqrt{(\lambda x_1)^2+(\lambda x_2)^2}} e^{-(w^2/((\lambda x_1)^2+(\lambda x_2)^2))} dw$$

Substituting $\frac{w}{\lambda} = v$,

$$f(\lambda x_1, \lambda x_2) = \lambda \int_0^{\sqrt{x_1^2+x_2^2}} e^{-(v^2/(x_1^2+x_2^2))} dv = \lambda f(x_1, x_2)$$

(Calculating range of v from range of $w : 0 \leq w \leq \lambda\sqrt{x_1^2+x_2^2} \iff 0 \leq \lambda v \leq \lambda\sqrt{x_1^2+x_2^2} \iff 0 \leq v \leq \sqrt{x_1^2+x_2^2}$)

12. The number of ways in which a beggar can be given at least 1 rupee from four 25-p coins, three 50-p coins and 2 one-rupee coins.
Total number of ways in which a beggar can be given money is (including the case when he is not given any money) $= 5 \times 4 \times 3 = 60$ (There are 5 ways in which you can give 25-p coins (give him none of it, give 1 coin, give 2 coins, give 3 coins, give 4 coins), similarly for 50-p and one-rupee coins). The number of ways in which he is given at least 1 rupee is the number of ways he can be given money minus the number of ways he is given at most 75 paise. He is given at most 75 paise in 6 ways (give him nothing, give him one 25-p coin, give him two 25-p coins, give him one 50-p coin, give him three 25-p coins, give him one 25-p and one 50-p coin). So, the total number of ways in which a beggar can be given at least 1 rupee is $60 - 6 = 54$.

13. A randomly chosen group is tested for a disease. Within this group, each individual has a probability of 0.1 of having the disease. A test is performed to identify the individuals with the disease. The test has two outcomes: positive or negative. If the individual does not have the disease, the test outcome is "negative" 90 percent of the time. If the individual has the disease, the test outcome is "negative" 20 percent of the time. Individuals who test "positive" are sent to a hospital for further treatment. What is the probability that an individual sent to the hospital indeed has the disease?
$\Pr(\text{disease}) = 0.1$
$\Pr(\text{negative}|\text{no disease}) = 0.9$
$\Pr(\text{negative}|\text{disease}) = 0.2$
$\Pr(\text{disease}|\text{positive}) = \frac{\Pr(\text{disease, positive})}{\Pr(\text{positive})} = \frac{\Pr(\text{positive}|\text{disease})\Pr(\text{disease})}{\Pr(\text{positive, disease}) + \Pr(\text{positive, no disease})}$
$= \frac{\Pr(\text{positive}|\text{disease})\Pr(\text{disease})}{\Pr(\text{positive}|\text{disease})\Pr(\text{disease}) + \Pr(\text{positive}|\text{no disease})\Pr(\text{no disease})}$
$= \frac{0.8 \times 0.1}{(0.8 \times 0.1) + (0.1 \times 0.9)} = \frac{8}{17}$

14. Let the function $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be such that $f(1) = 3$ and $f'(1) = 9$, where $\mathbb{R}_{++}$ is the positive part of the real line. Then $\lim_{x\to 0}\left(\frac{f(1+x)}{f(1)}\right)^{\frac{1}{x}}$ equals

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{f(1+x)}{f(1)}\right)^{\frac{1}{x}} &= \lim_{x\to 0} e^{\frac{1}{x}\log\left(\frac{f(1+x)}{f(1)}\right)} \\
&= e^{\lim_{x\to 0}\frac{1}{x}\log\left(\frac{f(1+x)}{f(1)}\right)} \ldots \text{by continuity of exponential function} \\
\lim_{x\to 0}\frac{1}{x}\log\left(\frac{f(1+x)}{f(1)}\right) &= \lim_{x\to 0}\frac{f'(1+x)}{f(1+x)} \ldots \text{by L'Hospital's Rule} \\
&= 3 \\
\text{Thus, } \lim_{x\to 0}\left(\frac{f(1+x)}{f(1)}\right)^{\frac{1}{x}} &= e^3
\end{aligned}$$

15. Consider the two variable linear regression model where error U_i are iid with mean 0 and variance 1.

$$Y_i = A + B(X_i - \bar{X}) + U_i$$

i is $1, 2, 3, \ldots n$

Let a and b be OLS estimates of A and B. Find correlation coefficient between a and b.

$$\text{Cor}(a, b) = \text{Cov}(a, b)/\sqrt{\text{Var}(a)\text{Var}(b)}$$

Now OLS estimates of A and B are

$$\begin{aligned} a &= \frac{\sum_{i=1}^n Y_i}{n} \\ b &= \frac{\sum_{i=1}^n Y_i(X_i - \bar{X})}{\sum_{i=1}^n (X_i - \bar{X})^2} \end{aligned}$$

Using $Y_i = A + B(X_i - \bar{X}) + U_i$ we can rewrite a and b as

$$\begin{aligned} a &= \frac{\sum_{i=1}^n A + B(X_i - \bar{X}) + U_i}{n} = A + \frac{\sum_{i=1}^n U_i}{n} \\ b &= \frac{\sum_{i=1}^n (A + B(X_i - \bar{X}) + U_i)(X_i - \bar{X})}{\sum_{i=1}^n (X_i - \bar{X})^2} = B + \frac{\sum_{i=1}^n U_i(X_i - \bar{X})}{\sum_{i=1}^n (X_i - \bar{X})^2} \end{aligned}$$

Therefore, $E(a) = A$ and $E(b) = B$ because $E(U_i) = 0$, and X_is are non-stochastic.

$$\begin{aligned} \text{Cov}(a, b) &= E((a - E(a))(b - E(b))) \\ &= E(\frac{\sum_{i=1}^n U_i}{n} \frac{\sum_{i=1}^n U_i(X_i - \bar{X})}{\sum_{i=1}^n (X_i - \bar{X})^2}) \\ &= \frac{1}{n\sum_{i=1}^n (X_i - \bar{X})^2} E(\sum_{i=1}^n \sum_{j=1}^n U_i U_j (X_j - \bar{X})) \\ &= \frac{1}{n\sum_{i=1}^n (X_i - \bar{X})^2} \sum_{i=1}^n \sum_{j=1}^n E(U_i U_j (X_j - \bar{X})) \\ &= \frac{1}{n\sum_{i=1}^n (X_i - \bar{X})^2} \sum_{i=1}^n \sum_{j=1}^n (X_j - \bar{X}) E(U_i U_j) \\ &= \frac{1}{n\sum_{i=1}^n (X_i - \bar{X})^2} \sum_{i=1}^n (X_i - \bar{X}) E(U_i^2) \\ & \quad [E(U_i U_j) = 0 \text{ for } i \neq j \text{ because } U_i\text{s are iid with } E(U_i) = 0] \\ &= \frac{1}{n\sum_{i=1}^n (X_i - \bar{X})^2} \sum_{i=1}^n (X_i - \bar{X}) \\ & \quad [\text{because } E(U_i^2) = 1] \\ &= 0 \end{aligned}$$

Therefore, $\text{Cor}(a, b) = 0$.

16. In a rectangular array (matrix) of distinct positive numbers, which has m rows and n columns, let x denote the largest of the smallest number in each column, and y the smallest of the largest number in each row. Then one can infer:

   (a) $x \geq y$
   (b) $y \geq x$
   (c) $x = y$
   (d) none of the above

Let a_{ij} denote the element in the ith row and jth column of the matrix where $1 \leq i \leq m$ and $1 \leq j \leq n$. Let x denote the largest of the smallest number in each column i.e.,

$$x = \max_{1 \leq j \leq n} [\min_{1 \leq i \leq m} a_{ij}]$$

y denote the smallest of the largest number in each row i.e.,

$$y = \min_{1 \leq i \leq m} [\max_{1 \leq j \leq n} a_{ij}]$$

Now, Smallest number in a column is smaller than every number in that column. Given j,

$$\min_{1\leq i\leq m} a_{ij} \leq a_{ij} \forall 1 \leq i \leq m$$

Therefore, the row produced by collecting the smallest numbers from each column will have a smaller maximum than maximum of any row in the matrix.

$$x = \max_{1\leq j\leq n} \min_{1\leq i\leq m} a_{ij} \leq \max_{1\leq j\leq n} a_{ij} \forall 1 \leq i \leq m$$

Now x is smaller than maximum of every row, hence it is smaller than the smallest of them.

$$x = \max_{1\leq j\leq n} \min_{1\leq i\leq m} a_{ij} \leq \min_{1\leq i\leq m} \max_{1\leq j\leq n} a_{ij} = y$$

17. Evaluate

$$\lim_{n\to\infty} \frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n-1}}{n^{3/2}}$$

Now,

$$\frac{\int_0^{n-1}\sqrt{x}dx}{n^{3/2}} \leq \frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n-1}}{n^{3/2}} \leq \frac{\int_1^{n}\sqrt{x}dx}{n^{3/2}} \quad \forall n \quad \text{(Think why?)}$$

Taking limits,

$$\lim_{n\to\infty}\frac{\int_0^{n-1}\sqrt{x}dx}{n^{3/2}} \leq \lim_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n-1}}{n^{3/2}} \leq \lim_{n\to\infty}\frac{\int_1^{n}\sqrt{x}dx}{n^{3/2}}$$

Now Applying L'Hopital's rule,

$$\lim_{n\to\infty}\frac{\int_0^{n-1}\sqrt{x}dx}{n^{3/2}} = \lim_{n\to\infty}\frac{\sqrt{n-1}}{\frac{3}{2}\sqrt{n}} = \lim_{n\to\infty}\frac{2}{3}\sqrt{1-\frac{1}{n}} = \frac{2}{3}$$

And

$$\lim_{n\to\infty}\frac{\int_1^{n}\sqrt{x}dx}{n^{3/2}} = \lim_{n\to\infty}\frac{\sqrt{n}}{\frac{3}{2}\sqrt{n}} = \frac{2}{3}$$

Hence,

$$\lim_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n-1}}{n^{3/2}} = \frac{2}{3}$$

18. (ME-1.13(2004))

$$\begin{aligned}
u &= \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)\\
\sin u &= \left(\frac{x^2+y^2}{x+y}\right)\\
\cos u\frac{\partial u}{\partial x} &= \frac{2x(x+y)-(x^2+y^2)}{(x+y)^2} = \frac{x^2-y^2+2xy}{(x+y)^2}\\
\cos u\frac{\partial u}{\partial y} &= \frac{2y(x+y)-(x^2+y^2)}{(x+y)^2} = \frac{y^2-x^2+2xy}{(x+y)^2}\\
x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} &= \frac{x^3-xy^2+2x^2y+y^3-yx^2+2xy^2}{(x+y)^2}\\
&= \frac{x^3+x^2y+y^3+xy^2}{(x+y)^2}\\
&= \frac{x^2(x+y)+y^2(x+y)}{(x+y)^2}\\
&= \frac{x^2+y^2}{x+y}\\
&= \sin u\\
\text{Hence, } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= \frac{\sin u}{\cos u} = \tan u
\end{aligned}$$

19. (ME-1.16(2009)) Given $n \geq 9$, $\mu = n^{\frac{1}{2}} + n^{\frac{1}{3}} + n^{\frac{1}{4}}$.
Clearly at $n = 9$, $\mu < n$ because $n^{\frac{1}{4}} < n^{\frac{1}{3}} < n^{\frac{1}{2}} = 3$.

$$\frac{d\mu}{dn} = \frac{1}{2}n^{\frac{-1}{2}} + \frac{1}{3}n^{\frac{-2}{3}} + \frac{1}{4}n^{\frac{-3}{4}} < \frac{1}{2}n^{\frac{-1}{2}} + \frac{1}{2}n^{\frac{-1}{2}} + \frac{1}{2}n^{\frac{-1}{2}} \leq \frac{1}{2} \quad \forall n \geq 9$$

Since $\frac{d\mu}{dn} < 1$ at all $n \geq 9$ and at $n = 9$, $\mu < 9$ therefore, $\mu < n \ \forall n \geq 9$.

20. Suppose .5 units of good 2 are required in the production of one unit of good 1 and .5 units of good 1 are required in the production of good 2.

    (a) What is the input-output matrix A for this economy?

    (b) What is the total production vector $x = (x_1, x_2)$ required for the economy to supply consumption vector $y = (3, 6)$ to consumers?

21. A price taking firm makes machine tools Y using labour and capital according to the following production function

$$Y = K^{0.25}L^{0.25}$$

Labour can be hired at the beginning of every week, while capital can be hired only at the beginning of every month. It is given that the wage rate = rental rate of capital = 10. Show that the short run (week) cost function is $10Y^4/K^*$ where the amount of capital is fixed at K^* and the long run (month) cost function is $20Y^2$.