Delhi School of Economics
Department of Economics

Entrance Examination for M. A. Economics
Option B
June 27, 2009

Time 3 hours **Maximum marks** 100

General instructions. Please read the following instructions carefully.

• Check that your examination has pages 1 to 6 and you have been given a blank Answer booklet. Do **not** start writing until instructed to do so by the invigilator.

• Fill in your Name and Roll Number on the small slip attached to the Answer booklet. Do **not** write this information anywhere else in this booklet.

• When you finish, hand in this Examination along with the Answer booklet to the invigilator.

• Do not disturb your neighbors at any time. **Anyone engaging in illegal examination practices will be immediately evicted and that person's candidature will be cancelled.**

Do not write below this line.

This space is for official use only.

Fictitious Roll Number

Question	**Marks**
1	
2	
3	
4	
5	
6	
Total	

Section I

Instructions. *Answer Question 1.*

Question 1. Each of the ten parts ((A) to (J)) of this question is followed by four possible answers ((a) to (d)), one of which is correct. Indicate the correct answer for each part in your answer booklet. Each correct choice will earn you 2 marks. However, you will lose 2/3 mark for each incorrect choice.

(A) A binary relation R on a set S is a subset of S^2. R is said to be transitive if, for all $x, y, z \in S$, $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$. R is said to be negatively transitive if, for all $x, y, z \in S$, $(x, y) \notin R$ and $(y, z) \notin R$ implies $(x, z) \notin R$. Define the binary relation R^{-1} on S by $R^{-1} = \{(x, y) \in S^2 \mid (y, x) \in R\}$. Consider the following statements:

- If R is transitive, then it is not negatively transitive.
- If R is transitive, then R^{-1} is transitive.
- If R is transitive, then R^{-1} is negatively transitive.

How many of the above statements are true?

(a) 0
(b) 1
(c) 2
(d) 3

(B) Consider a language with only two letters in its alphabet, O and W. This language obeys the following rules: (i) Deleting successive letters WO from any word which has more than two letters yields another word with the same meaning. (ii) Inserting OW or $WWOO$ in any place in a word yields another word with the same meaning. (iii) O, $OWOOW$, WOO and OWW are words in this language. Which of the following statements is *false*?

(a) The words WOO and OWW have the same meaning.
(b) WOO and OWW may not have the same meaning.
(c) O and $OWOOW$ have the same meaning.
(d) Both (b) and (c) are true.

(C) There are three persons: A, B and C. One of them is a Truth-teller (who always tells the truth), another is a Liar (who always lies) and the third is a normal person (who sometimes lies, other times speaks the truth). A said: "I am a normal person." B said: "A and C sometimes tell the truth." C said: "B is a normal person." A, B and C are aware of every person's nature.

(a) These statements are insufficient to determine the Liar.

(b) A is the normal person, B is the Truth-teller, C is the Liar.

(c) These statements are insufficient to determine the Liar, the normal person and the Truth-teller.

(d) A is the Liar, B is the normal person, C is the Truth-teller.

(D) Let $f(c) = \max\{x + 2y \mid x \geq 0,\ y \geq 0,\ 2x + y = c\}$. The derivative of f at c is

(a) c

(b) 0

(c) 2

(d) $c/2$

(E) Consider the statements: (i) $5^{44} > 4^{53}$, and (ii) $2^{100} + 3^{100} < 4^{100}$

(a) Both (i) and (ii) are false.

(b) (i) is true, (ii) is false.

(c) Both (i) and (ii) are true.

(d) (i) is false, (ii) is true.

(F) Let Y denote the number of heads obtained when three fair coins are tossed. The variance of Y^2 is

(a) 9.5

(b) 8.5

(c) 6.5

(d) 7.5

(G) Consider events A and B with $\text{Prob}(A) = 0.4$ and $\text{Prob}(B) = 0.7$. The maximum and minimum values of $\text{Prob}(A \cap B)$ respectively are

(a) 0.4 and 0.1

(b) 0.7 and 0.4

(c) 0.7 and 0.1

(d) 0.4 and 0

(H) The nine digits $1, \ldots, 9$ are arranged randomly to form a nine digit number with each digit used exactly once. The probability that 1, 2 and 3 appear as neighbors in the increasing order is

(a) 1/12

(b) 1/72

(c) 1/84

(d) $(2/3)^9$

(I) A blood test detects a given disease with probability 8/10 given that the tested person actually has the disease. With probability 2/10, the test incorrectly shows the presence of the disease in a disease-free person. Suppose 1/10 of the population has the disease. What is the probability that the person tested actually has the disease if the test indicates the presence of the disease?

(a) 1
(b) 9/13
(c) 4/13
(d) 7/13

(J) Suppose X and Y are independent random variables with standard Normal distributions. The probability of $X < -1$ is some $p \in (0, 1)$. What is the probability of the event: $X^2 > 1$ and $Y^3 < -1$?

(a) $3p$
(b) p^2
(c) $2p^2$
(d) $3p^2$

Section II

Instructions. *Answer any four of the following five questions in the Answer booklet. Each question is worth 20 marks. The marks for various parts of a question are indicated at the end of each question.*

Question 2. Let $\langle .,. \rangle : \Re^n \times \Re^n \to \Re$ be an inner product on $\Re^n$.

(A) Show that, if $x, y \in \Re^n$, then $\langle x, y\rangle \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}$.

(B) Show that equality holds in (A) if and only if $y = 0$ or $x = \lambda y$ for some $\lambda \in \Re$.

Let A be an $n \times n$ real matrix. A subspace V of $\Re^n$ is said to be invariant with respect to A if $AV \subset V$.

(C) If A is symmetric and $V \subset \Re^n$ is an invariant subspace with respect to A, then there exists $\lambda \geq 0$ and $x \in V$, $x \neq 0$, such that $A^2 x = \lambda x$.

$(6, 2, 12)$

Question 3. Let V be a vector space and let $\mathcal{L}(V, V)$ be the set of all linear transformations $T : V \to V$. Let $I \in \mathcal{L}(V, V)$ be the identity transformation. $P \in \mathcal{L}(V, V)$ is called a projector of V if (a) $V = \mathcal{R}(P) \oplus \mathcal{N}(P)$, and (b) $P(u + w) = u$ for all $u \in \mathcal{R}(P)$ and

$w \in \mathcal{N}(P)$; where $\mathcal{R}(P)$ denotes the range space of P, $\mathcal{N}(P)$ denotes the null space of P and $\oplus$ denotes a direct sum. Prove the following statements.

(A) P is a projector of V if and only if it is idempotent, i.e., $P^2 = P$.

(B) If U is a vector space and $X : U \to V$ is a linear transformation with $\mathcal{R}(P) = \mathcal{R}(X)$, then P is a projector if and only if $PX = X$.

(C) P is a projector if and only if $I - P$ is a projector.

$(8, 6, 6)$

Question 4. Consider a metric space (X, d) and a function $f : X \to \Re$. f is said to be upper semicontinuous at $x \in X$ if for every $\epsilon > 0$, there exists an open neighborhood U of x, such that $f(U) \subset (-\infty, f(x) + \epsilon)$. f is said to be upper semicontinuous if it is so at every $x \in X$. Prove the following statements.

(A) f is upper semicontinuous if and only if $f^{-1}((-\infty, r))$ is open in (X, d) for every $r \in \Re$.

(B) If $\{f_i \mid i \in I\}$ is a family of upper semicontinuous functions $f_i : X \to \Re$, then the function $g : X \to \bar{\Re}$ defined by $g(x) = \inf\{f_i(x) \mid i \in I\}$ is upper semicontinuous.

(C) If $\{f_i \mid i \in I\}$ is a family of upper semicontinuous functions $f_i : X \to \Re$ and I is a finite set, then the function $g : X \to \bar{\Re}$ defined by $g(x) = \sup\{f_i(x) \mid i \in I\}$ is upper semicontinuous.

$(6, 6, 8)$

Question 5. Consider the Euclidean metric space $(\Re^n, \|.\|)$. Let $X \subset \Re^n$ and $f : X \to \Re$. X is said to be a convex set if for every $x, y \in X$ and $t \in (0, 1)$, we have $tx + (1-t)y \in X$. f is said to be a convex function at $x \in X$ if for every $y \in X$ and $t \in (0, 1)$, $tx + (1-t)y \in X$ implies $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$. f is said to be a convex function if it is a convex function at every $x \in X$. Prove the following statements.

(A) If X is a convex set, then $f : X \to \Re$ is a convex function if and only if $\{(x, y) \in X \times \Re \mid f(x) \leq y\}$ is a convex set.

(B) If X is open in $\Re^n$ and f is convex and differentiable at $x \in X$, then

$$f(y) - f(x) \geq Df(x).(y - x)$$

for every $y \in X$, where $Df(x)$ denotes the derivative of f at x.

(C) If X is open in $\Re^n$ and f is convex and twice differentiable at $x \in X$, then $D^2 f(x)$ is positive semidefinite.

$(4, 8, 8)$

Question 6. Let $\mathcal{N}$ be the set of natural numbers. A set X is said to be finite if it is empty or there exists a bijection $f : \{1, \ldots, n\} \to X$ for some $n \in \mathcal{N}$. A set X is said to be denumerable if there is a bijection $f : \mathcal{N} \to X$. If X is either finite or denumerable, then it is said to be countable. Prove the following statements.

(A) Consider an indexed family of sets $\{X_i \mid i \in I\}$. If I is denumerable and X_i is denumerable for every $i \in I$, then $\cup_{i \in I} X_i$ is denumerable.

(B) If $n \in \mathcal{N}$ and X is a denumerable set, then X^n is denumerable.

(C) A set X is countable if and only if there is an injection $f : X \to \mathcal{N}$.

$(6, 7, 7)$