Delhi School of Economics
Department of Economics

Entrance Examination for M. A. Economics
Option B
June 26, 2010

Time. 3 hours **Maximum marks.** 100

Instructions. Please read the following instructions carefully.

- Do **not** break the seal on this booklet until instructed to do so by the invigilator. Anyone breaking the seal prematurely will be evicted from the examination hall and his/her candidature will be cancelled.
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- When you finish, hand in this examination booklet to the invigilator.
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Marks tally

Part	Marks
I	
II.11	
II.12	
II.13	
II.14	
II.15	
Total	

Part I

Instructions.

• Check that this examination booklet has pages 1 through 22. Also check that the bottom of each page is marked with *EEE 2010 B*.

• This part of the examination consists of 10 multiple-choice questions. Each question is followed by four possible answers, at least one of which is correct. If more than one choice is correct, choose **only the best one**. Among the correct answers, the best answer is the one that implies (or includes) the other correct answer(s). **Indicate your chosen best answer by circling the appropriate choice.**

• For each question, you will get 2 marks if you choose only the best answer. If you choose none of the answers, then you will get 0 for that question. **However, if you choose something other than the best answer or multiple answers, then you will get −2/3 mark for that question.**

You may begin now. Good luck!

QUESTION 1. Suppose the function $f : \Re^2 \to \Re$ is increasing in both arguments, i.e., $f(x, y)$ is increasing in x and increasing y. For $x, y \in \Re$, let

$$x \wedge y = \begin{cases} x, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$$

Define $g : \Re^2 \to \Re$ by

$$g(x, y) = \begin{cases} f(x, y) - \frac{1}{2} f(x \wedge y, x \wedge y), & \text{if } x \geq y \\ \frac{1}{2} f(x \wedge y, x \wedge y), & \text{if } x < y \end{cases}$$

Which of the following statements is correct?

(a) g is increasing in x and decreasing in y

(b) g is increasing in both x and y

(c) g is increasing in x but may or may not be increasing in y

(d) g may or may not be increasing in x

QUESTION 2. Suppose the function $f : \Re \to \Re$ is given by $f(x) = x^3 - 3x + b$. Find the number of points in the closed interval $[-1, 1]$ at which $f(x) = 0$.

(a) None

(b) At most one

(c) One

(d) At least one

QUESTION 3. Consider a twice differentiable function $f : \Re \to \Re$ and $a, b \in \Re$ such that $a < b$, $f(a) = 0 = f(b)$ and $D^2 f(x) + Df(x) - 1 = 0$ for every $x \in [a, b]$. Then,

(a) $f(x) \leq 0$ for every $x \in [a, b]$
(b) $f(x) \geq 0$ for every $x \in [a, b]$
(c) $f(x) = 0$ for every $x \in [a, b]$
(d) f must take positive and negative values on the interval $[a, b]$

QUESTION 4. Suppose X_1 and X_2 are real-valued random variables with f as their common probability density function. Suppose (x_1, x_2) is a sample generated by these random variables. The expectation of the number of observations in the sample that fall within a specified interval $[a, b]$ is

(a) $\left(\int_a^b f(x)\,dx\right)^2$
(b) $\int_a^b x^2 f(x)\,dx$
(c) $2\int_a^b f(x)\,dx$
(d) $\int_a^b x f(x)\,dx$

QUESTION 5. Suppose $X_1, \ldots, X_n$ are observed completion times of an experiment with values in $[0, 1]$. Each of these random variables is uniformly distributed on $[0, 1]$. If Y is the maximum observed completion time, then the mean of Y is

(a) $[n/(n+1)]^2$
(b) $n/2(n+1)$
(c) $n/(n+1)$
(d) $2n/(n+1)$

QUESTION 6. Suppose player 1 has five coins and player 2 has four coins. Both players toss all their coins and observe the number that come up heads. Assuming all the coins are fair, what is the probability that player 1 obtains more heads than player 2?

(a) 1/2
(b) 4/9
(c) 5/9
(d) 4/5

QUESTION 7. Suppose θ is a random variable with uniform distribution on the interval $[-\pi/2, \pi/2]$. The value of the distribution function of the random variable $X = \sin\theta$ at $x \in [-1, 1]$ is

(a) $\sin^{-1}(x)$
(b) $\sin^{-1}(x) + \pi/2$

(c) $\sin^{-1}(x)/\pi + 1/2$
(d) $\sin^{-1}(x)/\pi + \pi/2$

QUESTION 8. Let X be a normally distributed random variable with mean 0 and variance σ^2. Then, the mean of X^2 is

(a) 0
(b) σ
(c) 2σ
(d) σ^2

The next two questions are based on the following data. *The number of loaves of bread sold by a bakery in a day is a random variable X. The distribution of X has a probability density function f given by*

$$f(x) = \begin{cases} kx, & \text{if } x \in [0,5) \\ k(10-x), & \text{if } x \in [5,10) \\ 0, & \text{if } x \in [10,\infty) \end{cases}$$

QUESTION 9. As f is a probability density function, the value of k must be

(a) 0
(b) $-2/25$
(c) $1/25$
(d) $2/75$

QUESTION 10. Let A be the event that $X \geq 5$ and let B be the event that $X \in [3,8]$. The probability of A conditional on B is

(a) 16/37
(b) 21/37
(c) 25/37
(d) 1

End of Part I.

Proceed to Part II of the examination on the next page.

Part II

Instructions.

- Answer **any four** out of Questions 11, 12, 13, 14 and 15 in the space following the relevant question. **No other paper will be provided for this purpose.** You may use the blank pages at the end of this booklet, marked **Rough work**, to do rough calculations, drawings, etc. **However, your "Rough work" will not be read or checked.**
- Each question is worth 20 marks.

QUESTION 11. Let $G : \Re \to [0,1]$ be a non-decreasing, right continuous function with $G(x) = 0$ for every $x < 0$ and $G(x) = 1$ for every $x \geq 1$. Define $b : [0,1] \to \Re$ by

$$b(c) = \begin{cases} 0, & \text{if } c = 0 \\ \inf G^{-1}([c,1]), & \text{if } c \in (0,1] \end{cases}$$

(Notation: inf A denotes the infimum, or greatest lower bound, of A.)

(A) Show that $G(b(c)) \geq c$ for every $c \in [0,1]$. Also show that b is nonnegative, increasing, bounded and left continuous.

(B) Show that b is continuous if and only if G is strictly increasing on $[0,1]$.

(C) Show that G is continuous if and only if b is strictly increasing on $[0,1]$.

Answer.

QUESTION 12. Consider an open set $C \subset \Re$ and $f : C \to \Re$. C is said to be a **convex set** if $x, y \in C$ and $t \in (0,1)$ implies $tx + (1-t)y \in C$. f is said to be a **convex function** if $x, y \in C$ and $t \in (0,1)$ implies $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$. f is said to **locally bounded** if, for every $x \in C$, there exists $r > 0$ and $M \in \Re_+$, such that $|f(y)| \leq M$ for every $y \in B_r(x)$.

(A) Suppose C is convex and f is convex. Show that f is locally bounded.

(Hint: Find $r > 0$ such that $B_r(x) \subset C$. Consider $y \in B_r(x)$. Use the convexity of f to get an upper bound on $f(y)$ that is independent of y, say M. Noting that there exists $z \in B_r(x)$ such that $x = y/2 + z/2$, use the convexity of f to get a lower bound on $f(y)$ that is independent of y.)

(B) By (A), there exists $r > 0$ and $M \in \Re_+$ such that $B_{2r}(x) \subset C$ and $|f(y)| \leq M$ for every $y \in B_{2r}(x)$. Consider distinct $y, z \in B_r(x)$ and set $w = z + (r/\alpha)(z - y)$, where $\alpha = |y - z|$. Show that $w \in B_{2r}(x)$ and that z is a convex combination (i.e., weighted average) of y and w.

(C) Use the convexity of f to show that

$$f(z) - f(y) \leq \frac{\alpha}{r}|f(w) - f(y)| \leq \frac{2M}{r}|z - y|$$

(D) Interchanging the roles of y and z in (B) and (C), we have $|f(z) - f(y)| \leq (2M/r)|z - y|$. Show that f is continuous.

Answer.

QUESTION 13. (Notation: $D\phi$ denotes the derivative of a function ϕ.)

(A) A function $f : \Re^n \to \Re$ is said to be homogeneous of degree $k > 0$ if $f(tx) = t^k f(x)$ for every $x \in \Re^n$ and $t > 0$. Given $f : \Re^n \to \Re$ and $k > 0$, show that f is homogeneous of degree k if and only if $Df(x).x = kf(x)$ for every $x \in \Re^n$.

(B) Consider a pair of time-dependent variables $k : \Re_+ \to \Re_+$ and $q : \Re_+ \to \Re_+$ whose values at time t satisfy the pair of ODEs:

$$Dk(t) = f(k(t)) - nk(t)$$

and

$$Dq(t) = q(t)[(n + \rho) - Df(k(t))]$$

Suppose $n > 0$, $\rho > 0$ and $f : \Re \to \Re$ is twice continuously differentiable such that $f(0) = 0$, $Df(0) \in (n + \rho, \infty)$ and $\lim_{k\uparrow\infty} Df(k) = 0$. Moreover, suppose $Df(k) > 0$ and $D^2 f(k) < 0$ for every $k \in \Re$, and there exists $k > 0$ such that $f(k) > nk$.

Show the existence of k^* such that $Df(k^*) = n + \rho$. Show the existence of k^{**} such that $f(k^{**}) = nk^{**}$. Show that $k^{**} > k^*$. Using these facts, draw the phase diagram for the above pair of ODEs. Analyze the dynamics of (k, q) in this diagram if $k(0) \geq 0$ and $q(0) > 1$.

Answer.

QUESTION 14. Let U be a finite dimensional vector space with inner product $\langle .,. \rangle$. Given a vector subspace S of U, let $\mathcal{O}(S) = \cap_{x \in S}\{y \in U \mid \langle x, y\rangle = 0\}$.

(A) Show that $U = S \oplus \mathcal{O}(S)$.

Let V and W be finite dimensional vector spaces with inner products $\langle .,. \rangle^V$ and $\langle .,. \rangle^W$ respectively. Let $\mathcal{L}(V, W)$ be the space of linear transformations from V to W. Given $A \in \mathcal{L}(V, W)$, let $\mathcal{R}(A)$ be the range space of A and let $\mathcal{N}(A)$ be the null space of A. If $A \in \mathcal{L}(V, W)$, then $A^* \in \mathcal{L}(W, V)$ is called the adjoint of A if $\langle x, A^*y\rangle^V = \langle Ax, y\rangle^W$ for all $x \in V$ and $y \in W$.

Prove the following propositions for $A \in \mathcal{L}(V, W)$.

(B) $\mathcal{N}(A) = \mathcal{O}(\mathcal{R}(A^*))$ and $\mathcal{R}(A) = \mathcal{O}(\mathcal{N}(A^*))$.

(C) A and A^* have the same rank.

(D) A is injective if and only if A^* is surjective.

(E) $\mathcal{R}(A^*A) = \mathcal{R}(A^*)$ and $\mathcal{N}(A^*A) = \mathcal{N}(A)$.

Answer.

QUESTION 15. (A) You are playing heads (H) or tails (T) with Prosser but you suspect that his coin is unfair. J. von Neumann suggested that you proceed as follows: toss Prosser's coin twice. If the outcome is HT, then call the result "win". If it is TH, then call the result "lose". If it is TT or HH, then ignore the outcome and toss Prosser's coin twice again. Follow this procedure until you get either an HT (win) or a TH (lose). Show that the probability of winning is 1/2.

(B) In order to estimate the volume of a cube, the length of one side is sampled n times. The sampled values are denoted $x_1, \ldots, x_n$. The measurements are assumed to be normally distributed with mean μ (the true length of the cube) and variance σ^2. Two estimates of the volume are considered, namely $m_1 = n^{-1} \sum_{i=1}^{n} x_i^3$ and $m_2 = (n^{-1} \sum_{i=1}^{n} x_i)^3$. Which of these estimators has a smaller bias? What is an unbiased estimator of μ^3?

Answer.