Group A (Microeconomics)

1. [30 marks: $4 + 20 + 6$]

   Consider a used car market with 600 buyers each willing to buy exactly one used car, and 500 sellers each having exactly one used car. Out of the 500 used cars, 400 are of good quality (*peaches*) and 100 are of bad quality (*lemons*). The monetary valuation of owning a peach is Rs. 100 for a buyer and Rs. 50 for a seller. On the other hand, the monetary valuation of owning a lemon is Rs. 10 for both a buyer and a seller. A seller knows whether the car she owns is a peach or a lemon, whereas a buyer only knows that there are 400 peaches and 100 lemons. Both the buyers and the sellers know the various valuations.

(a) What outcome maximizes the aggregate surplus of the economy? Provide a clear explanation for your answer.

(b) (i) Derive, with a clear explanation, the supply of used cars as a function of price. Draw this supply curve by plotting number of used cars on x-axis and price on y-axis. [You must label all the important points in the figure clearly.]

(ii) Derive, with a clear explanation, the demand for used cars as a function of price. Draw this demand curve in the same figure as in part (i). [You must label all the important points in the figure clearly.]

(iii) Use the demand and supply functions above to find out *all possible* competitive equilibria in the used car market mentioning clearly which types of car, lemon or peach, are bought and sold in each equilibrium.

(c) Now suppose that buyers also know the identity of all cars, that is, whether any given car is a peach or a lemon. Use a similar demand-supply analysis as above to solve for *all possible* competitive equilibria in the used car market in this scenario.

2. [30 marks: $5 + 3 + 3 + 11 + 8$]

Consider an industry with 2 firms – a ***private*** firm (indexed by r) and a ***public*** firm (indexed by u) – producing a homogeneous product and competing in quantities. The firms face an inverse demand function $p = a - bQ$, $a > 0$, $b > 0$, where $Q = q_r + q_u$ denotes aggregate output, and q_r and q_u denote the amounts of output produced by the private and public firms respectively. Each firm i faces the total cost of production cq_i, $i = r, u$, $0 < c < a$.

(a) For any q_r and q_u, derive the expressions for (i) private firm's profit, (ii) public firm's profit, (iii) consumer surplus, and (iv) welfare (sum of consumer surplus and producer surplus).

(b) The *private* firm's objective is to maximize its *own profit.* For a given q_u, set up the private firm's maximization problem and derive its optimal choice of output q_r. [This exercise gives you the *reaction function* of the private firm.]

(c) The *public* firm's objective is to maximize *welfare*. For a given q_r, set up the public firm's maximization problem and derive its optimal choice of output q_u. [This exercise gives you the *reaction function* of the public firm.]

(d) Recall that the two firms compete in quantities.

(i) Define the concept of equilibrium in this context and find out the amounts of output, q_r^* and q_u^*, the two firms produce in equilibrium. Find out the expressions of price, profits of the two firms, consumer surplus and welfare in the equilibrium.

(ii) Illustrate this equilibrium by drawing the two reaction functions you have derived in parts (b) and (c) (plot q_u in x-axis and q_r in y-axis). [You must label the important points in the figure clearly.]

(e) Suppose that the marginal cost of the private firm falls to $c_r < c$ while the marginal cost of the public firm remains the same at c. Draw the new reaction functions and explain clearly how the following outcomes change in the new equilibrium (as compared to the old equilibrium): q_r, q_u, Q, price, profits of the two firms, consumer surplus and welfare. [There is no need to derive the exact expressions; just qualitative answers are enough.]

3. [30 marks: $6 + 4 + 20$]

There is a unit mass of consumers all of whom want to purchase at most 1 unit of a good. Consumer v has a valuation v for this good, where $v \in [0, 1]$. Assume that v is uniformly distributed over interval $[0, 1]$ so that the number of consumers with valuation in between a and b, where $0 \leq a < b \leq 1$, is $b - a$. There is a monopoly firm with total cost of producing q units of the good given by $\frac{q}{3}$. The firm does not know the identity of any consumer and hence must charge a uniform price to all the consumers.

(a) For any price p, derive, with a clear explanation, the demand facing the monopoly firm.

(b) Derive, with a clear explanation, the monopoly price and profit level.

(c) Suppose that the firm can, for a cost, get to know whether a consumer belongs to the interval $[0, \frac{4}{5}]$, or to the interval $(\frac{4}{5}, 1]$. What is the maximum amount the firm is willing to pay for this information? Give a clear explanation for your answer.

Group B (Macroeconomics)

1. [30 marks: $4+6+10+5+5$]

   Consider an aggregate demand and aggregate supply model where, in the short run, aggregate capital is fixed at the level $\overline{K}$. The aggregate demand curve, aggregate output (Y) demanded as a function of aggregate price level (P), is given by a standard downward-sloping curve. The aggregate supply curve, aggregate output (Y) supplied as a function of aggregate price level (P), is not standard, and the question leads you to derive the aggregate supply curve.

   The aggregate production function is linear in capital and labour (L): $Y = AL + \overline{K}$, $A > 0$. The labour union is very powerful and dictates the minimum aggregate *nominal* wage rate as $\overline{W}$. Each worker is endowed with one unit of labour which they supply inelastically if the producers offer the nominal wage $W > \overline{W}$. A worker does not supply any labour if $W < \overline{W}$. At $W = \overline{W}$, a worker is indifferent between supplying and not supplying her labour endowment. The number of workers available in the economy is fixed at $\overline{L}$.

(a) Derive, with a clear explanation, the aggregate labour supply (L^S) in this economy as a function of the aggregate nominal wage rate, W.

(b) Note that the marginal product of labour is constant, $A > 0$.

   (i) Derive, with a clear explanation, the aggregate labour demand (L^D) in this economy as a function of the real wage rate, $\frac{W}{P}$.

   (ii) Using your answer to part (i) above, derive the aggregate labour demand (L^D) in this economy as a function of the aggregate nominal wage rate, W.

(c) Choose an arbitrary aggregate price level, P, and draw the aggregate labour supply (L^S) and aggregate labour demand (L^D) curves, as functions of W, by plotting labour (L) on x-axis and nominal wage (W) on y-axis. Think about the labour market equilibrium for the arbitrary aggregate price level P that you have chosen.

Note that the equilibrium employment (L^*) in the economy depends on the arbitrary price level P that you choose. Derive, with a clear explanation, the equilibrium employment (L^*) as a function of aggregate price level P.

(d) Derive, with a clear explanation, aggregate output (Y) supplied as a function of aggregate price level (P). Draw this aggregate supply curve by plotting Y on x-axis and P on y-axis.

(e) Recall that the aggregate demand curve is given by a standard downward-sloping curve. Explain the effectiveness of the standard monetary and fiscal policies in this set up.

2. [30 marks: $14 + 16$]

Consider the following version of the Solow growth model. The aggregate output at time t, Y_t, depends on the aggregate capital stock (K_t) and aggregate labour force (L_t) in the following way:

$$Y_t = (K_t)^\alpha (L_t)^{1-\alpha}, \; 0 < \alpha < 1.$$

There is perfect competition in the factor market so that, in equilibrium, each factor is paid its marginal product and the total output is distributed to all the households in the form of wage earnings and interest earnings. Households save a proportion $0 < s < 1$ of their disposable income in every period. All household savings are invested which augment the capital stock over time. There is no depreciation of capital. Population and therefore the aggregate labour force grows at a constant rate $n > 0$.

(a) The government taxes the *interest earnings* at the rate $0 < \tau < 1$. Wage earnings are not taxed. The government uses the collected taxes to fund *government consumption*; in particular, the tax collection is *not* used for investment at all.

(i) Derive, with clear explanations, the expressions for aggregate wage earning, aggregate interest earning and aggregate savings (S_t) of the economy in terms of Y_t.

(ii) Define $k_t \equiv \frac{K_t}{L_t}$, the capital-labour ratio in period t. Derive, with a clear explanation, the law of motion of capital-labour ratio, that is, the equation with k_{t+1} on the left-hand side and k_t on the right-hand side.

(iii) Derive, with a clear explanation, the steady-state level of capital-labour ratio in this economy, k^*, and examine how k^* changes with changes in the tax rate τ.

(b) As in part (a) above, the government continues taxing interest earnings at the rate τ and wage earnings are not taxed. But consider now that the tax revenue collected is used to fund investment by the government so that the capital stock is further augmented by this public investment.

(i) Derive, with a clear explanation, the expression for aggregate investment in this economy.

(ii) Derive, with a clear explanation, the new law of motion of capital-labour ratio.

(iii) Derive the new steady-state level of capital-labour ratio in this economy, k^{**}, and compare it with k^*. Does the comparison make economic sense?

(iv) How does k^{**} change with changes in τ? Compare with the response of k^{*} and explain the economic reason behind the differential impact.

3. [30 marks: $9 + 5 + 6 + 4 + 6$]

   Consider an individual who lives for two periods. In the first period, she earns an wage income W and takes her consumption-savings decision once income is realized. In the second period, she has no wage income but receives the return along with her principal amount of savings, s. The *gross* rate of interest is $R > 1$. Suppose that there is a government which collects an amount T in the form of lumpsum tax from the wage income in the first period (this can be considered as *mandatory savings* of the individual) and returns the amount T in the form of a lumpsum transfer in the second period. Suppose that the utility derived by the individual who consumes c_1 in the first period and c_2 in the second period is given by $u(c_1) + \beta u(c_2)$, where β is a discount factor with $0 < \beta < 1$ and the utility function u is strictly increasing and strictly concave.

(a) Set up the individual's utility maximization problem by specifying her budget constraint clearly. Derive the first-order condition of this utility maximization problem by showing your procedure clearly. Provide a clear economic interpretation of the first-order condition.

(b) Note that $\beta < 1$ implies that the individual is myopic (short-sighted), she puts less weight on future period. Explain intuitively whether a more myopic individual will save more or less than a less myopic individual. Verify your intuition by determining the sign of $\dfrac{ds}{d\beta}$.

(c) Note also that the individual's personal savings, s, depends on the government mandated savings T. Explain intuitively whether the government mandated savings increases or decreases personal savings. Verify your intuition by determining the sign of $\frac{ds}{dT}$.

(d) One rupee received in benefits in period 2 would require an individual to save an amount $\frac{1}{R}$ (< 1 since $R > 1$) in period 1. Explain intuitively whether the government mandated savings make the individual cut back her personal savings at a rate higher or lower than $\frac{1}{R}$. Verify your intuition.

(e) In the light of your answer to part (b) and from the expression of $\frac{ds}{dT}$ you expect that the rate of change in personal savings in response to a change in T depends on the discount factor β. Prove that more myopic individuals reduce their personal savings at a higher rate.

Group C (Mathematics)

1. [30 marks: $20 + 10$]

   Consider the following feasible region C in $\mathbb{R}^2$ for an optimization program:

$$C := \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, y \leq x\}.$$

(a) Suppose $a \neq 0, b \neq 0$. Show that an optimal solution to

$$\max_{(x,y)\in C} ax + by$$

is either $(0, 0), (1, 0), (1, 1)$. Describe all possible values of a and b for which each of $\{(0, 0), (1, 0), (1, 1)\}$ is an optimal solution.

(b) Suppose a and b are independently drawn from $[-1, 1]$ using a probability distribution with cumulative distribution function (cdf) F. What is the probability that the unique optimal solution to the above optimization problem is $(1, 0)$?

2. [30 marks: $5 + 10 + 15$]

   Suppose A, B, C, a, b, c are real numbers and $A \neq 0$, $a \neq 0$. Suppose for all real values of x, the following holds:

$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|.$$

Suppose $B^2 - 4AC > 0$.

(a) Argue that $|A| \geq |a|$.

(b) Show that $b^2 - 4ac > 0$.

(c) Show that $B^2 - 4AC \geq b^2 - 4ac$.

3. [30 marks: $5 + 5 + 15 + 5$]

   Let $\mathcal{F}$ be a class of functions from $[0, \infty)$ to $[0, \infty)$ with the following properties:

- The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in $\mathcal{F}$.
- If $f(x)$ and $g(x)$ (not necessarily distinct) are in $\mathcal{F}$, then the functions $f(x) + g(x)$ and $f(g(x))$ are in $\mathcal{F}$.
- If $f(x)$ and $g(x)$ are in $\mathcal{F}$ and $f(x) \geq g(x)$ for all $x \in [0, \infty)$, then $f(x) - g(x)$ is in $\mathcal{F}$.

(a) For any positive integer n, show that $h(x) = nx$ is in $\mathcal{F}$.

(b) If $f(x)$ and $g(x)$ are in $\mathcal{F}$, show that the functions $\ln(f(x) + 1)$ and $\ln(g(x) + 1)$ are in $\mathcal{F}$.

(c) If $f(x)$ and $g(x)$ are in $\mathcal{F}$, show that the function $f(x)g(x) + f(x) + g(x)$ is in $\mathcal{F}$.

(d) If $f(x)$ and $g(x)$ are in $\mathcal{F}$, show that the function $f(x)g(x)$ is in $\mathcal{F}$.